

Integral and Asymptotic Properties of Solitary Waves in Deep Water

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Abstract

We consider two- and three-dimensional gravity and gravity-capillary solitary water waves in infinite depth. Assuming algebraic decay rates for the free surface and velocity potential, we show that the velocity potential necessarily behaves like a dipole at infinity and obtain a related asymptotic formula for the free surface. We then prove an identity relating the “dipole moment” to the kinetic energy. This implies that the leading-order terms in the asymptotics are nonvanishing and in particular that the angular momentum is infinite. Lastly we prove that the “excess mass” vanishes. © 2018 the Authors. *Communications on Pure and Applied Mathematics* is published by the Courant Institute of Mathematical Sciences and Wiley Periodicals, Inc.

1 Introduction

We consider the motion of an infinitely deep fluid region under the influence of gravity that is bounded above by a free surface, including both *gravity waves*, where the pressure is constant along the free surface, and *gravity-capillary waves*, where it is proportional to the mean curvature. The fluid is assumed to be inviscid and incompressible, and the flow is assumed to be irrotational. We further restrict to traveling-wave solutions that appear steady in a moving reference frame and are *solitary* in that the free surface approaches some asymptotic height at infinity, normalized to zero.

Solitary waves in finite depth, where the fluid is instead bounded below by a flat bed, have a long and celebrated history; see, for instance, the reviews [16, 19, 33]. This includes a wide variety of existence results for gravity waves in two dimensions [4, 5, 17, 29, 32] and gravity-capillary waves in two [3, 8, 10, 24, 28] and three [9, 11, 20] dimensions. The two-dimensional gravity waves are *waves of elevation* in that their free surface elevations are everywhere positive [14], while some of the gravity-capillary waves are *waves of depression* with negative free surfaces, and still others have oscillatory free surfaces that change sign. For three-dimensional gravity waves, Craig [13] has ruled out waves of elevation or depression, while recent results for the time-dependent problem [39, 40] rule out sufficiently small solitary waves.

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In infinite depth, gravity solitary waves are conjectured not to exist, regardless of the dimension. Hur [22] has proved that the only two-dimensional solitary waves whose free surfaces are $O(1/|x|^{1+\epsilon})$ as $|x| \rightarrow \infty$ are *trivial waves* with flat free surfaces. For three-dimensional waves, Craig has shown, as in the finite-depth case, that there are no waves of elevation or depression [13]. Also, as in the case of finite depth, sufficiently small three-dimensional gravity waves are ruled out by global existence results [18, 42] for small data in the time-dependent problem.

Two-dimensional gravity-capillary waves in infinite depth were first rigorously constructed by Iooss and Kirrmann [25] following the pioneering numerical work of Longuet-Higgins [31] and Vanden-Broeck and Dias [38]. Their proof used normal form techniques; Buffoni [7] and Groves and Wahlén [21] have subsequently given variational constructions. One distinguishing feature of these solitary waves is their algebraic decay at infinity. In [25] the free surfaces were shown to be $O(1/|x|)$ as $x \rightarrow \infty$; Sun [36] later improved this to the expected $O(1/|x|^2)$. More generally, Sun proved that a $O(1/|x|^{1+\epsilon})$ free surface is automatically $O(1/|x|^2)$, and that in this case several integral identities hold. In particular, the “excess mass” vanishes so that no such wave can be a wave of elevation or depression.

While there are no rigorous existence results in the literature for three-dimensional gravity-capillary waves in infinite depth, we have been informed by Buffoni, Groves, and Wahlén that their finite-depth construction in [11] can be extended to infinite depth and moreover that there is an alternate construction in this case using the implicit function theorem. Three-dimensional capillary-gravity waves have also been calculated formally [27] and numerically [1, 34, 41]. Interestingly, as the amplitude of these waves approaches 0, their energy is predicted to approach a finite value [41, sec. 3.2.2]. This is consistent with recent global existence results for small data in the time-dependent problem [15], which rule out solitary waves that are small in a certain function space. We also mention that Hur [23] has recently generalized one of the integral identities in [36] to three (and higher) dimensions.

In this paper we simultaneously consider infinite-depth gravity and gravity-capillary solitary waves in dimension $n = 2$ or 3 . We assume (2.2) that the free surface is $O(1/|x|^{n-1+\epsilon})$ as $|x| \rightarrow \infty$ while the velocity potential is $o(1/|x|^{n-2})$. Our first conclusion is that the velocity potential behaves like a dipole near infinity (2.3), which implies related asymptotics (2.4) for the free surface. We next give an explicit formula (2.5) for the kinetic energy in terms of the “dipole moment” and the wave speed. For nontrivial waves, this ensures that the leading-order terms in our asymptotics are nonvanishing, which in turn implies that the angular momentum is infinite (Corollary 2.3). A modification of the proof of (2.5) shows that the “excess mass” vanishes (2.6).

We now briefly interpret our results in the context of the previous work mentioned above. The two-dimensional gravity waves we consider are automatically trivial by Hur’s nonexistence result [22], so that our results are interesting only in their method of proof. For three-dimensional gravity waves, on the other hand,

our results are entirely new. Our nonexistence proof for waves with finite angular momentum complements Craig's nonexistence result [13], which only applies to waves of elevation and depression, as well as the time-dependent results [18, 42], which only rule out waves that are sufficiently small. For two-dimensional gravity-capillary waves, we improve upon Sun's asymptotic bounds [36] by proving asymptotic formulas for both the free surface and velocity potential, with nonvanishing leading-order terms. Given this dipole-like behavior of the velocity potential, somewhat formal proofs of our integral identities were given in this case by Longuet-Higgins [31]. For three-dimensional capillary-gravity waves, even the conclusion that there are no waves of elevation or depression, which follows from either our asymptotics or from the vanishing of the excess mass, appears to be new. Under the assumption that the dipole moment and wave speed are parallel, our asymptotic formula (2.4) for the free surface is consistent with equation (5.6) in [27], which is derived as part of a small-amplitude expansion.

The two-dimensional results [22, 36] both use conformal mappings to obtain a problem in a half-plane. Hur goes on to write a nonlocal Babenko-type equation for the free surface elevation as a function of the velocity potential, while Sun exploits the existence of explicit Green's functions. In three dimensions, these arguments break down entirely. Instead of conformal mappings, we use the Kelvin transform. This does not fix the domain, but does convert questions about the asymptotic behavior of the velocity potential near infinity into questions about the regularity of the transformed potential near a finite point on the boundary of the transformed fluid domain. We obtain this boundary regularity using standard Schauder estimates for weak solutions to elliptic equations.

The existence of solitary waves with free surfaces that decay no faster than $1/|x|^{n-1}$ remains open. Fixing $n = 2$ for simplicity, we know only that such waves can be neither gravity waves of elevation or depression [13] nor gravity-capillary waves with $\sigma > |c|^2/4g$ [36]. The difficulty in studying these (purely hypothetical) solitary waves is the presence of periodic water waves in the same parameter regime. In Appendix B, we consider an explicit family of solitary waves with pressure forcing obtained by multiplying a linear periodic wave with a decaying envelope. The pressure forcing decays more quickly than the free surface, and the gap in decay rates strongly suggests that linear bootstrapping methods of the kind used in [14, 36] do not apply to waves decaying more slowly than $1/|x|$. Thus a new, presumably nonlinear technique is needed for further progress. This is not unlike the state of the art for two-dimensional gravity solitary waves in finite depth $h > 0$: We know only that a wave with subexponential decay is necessarily subcritical ($c^2 < gh$) and cannot be a wave of elevation or a monotone wave of depression [14, 26]. We note that here too the issue is the presence of periodic waves, and a family of forced examples suggests that new arguments are needed to treat surfaces that decay more slowly than $1/|x|$.

This paper is organized as follows. In Section 2, we state our main results. In Section 3, we prove Theorem 2.1 on the asymptotic behavior of the velocity potential and free surface near infinity. To streamline the presentation, some of the more technical details are deferred to Appendix A. In Section 4, we prove Theorem 2.2, which expresses the kinetic energy in terms of the wave speed and dipole moment, as well as its corollaries. The proof involves applying the divergence theorem to a carefully chosen vector field and then using the asymptotics from Theorem 2.1 to deal with one of the boundary terms. Finally, in Appendix B, we discuss several families of explicit and slowly decaying solitary waves with pressure forcing on the free surface, and consider the implications for the linear bootstrapping methods used by other authors.

Since a preprint of this paper first appeared, parts of it have been generalized by Chen, Walsh, and the author to waves that are not irrotational but instead have localized vorticity [12]. In particular, precise asymptotics at infinity are proven for the waves constructed by Shatah, Walsh, and Zeng in [35]. We also note that the decay assumptions in (2.2b) on derivatives of the free surface are weakened.

2 Results

There are two distinguished reference frames for a solitary wave: a moving frame where the motion appears steady, and another “lab” frame where the fluid velocity is assumed to vanish at infinity. As is common practice, we measure positions in the first frame and velocities in the second. We set $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $n = 2$ or 3 , with x' the horizontal coordinate and x_n the vertical coordinate. Assuming that the free surface S is a graph $x_n = \eta(x')$, the semi-infinite fluid domain is

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : x_n < \eta(x')\}.$$

Since the fluid is irrotational, the fluid velocity in the lab frame is the gradient of a velocity potential φ . The equations satisfied by φ and η are

$$(2.1a) \quad \Delta\varphi = 0 \quad \text{in } \Omega,$$

$$(2.1b) \quad \nabla\varphi \cdot N = c \cdot N \quad \text{on } S,$$

$$(2.1c) \quad \frac{1}{2}|\nabla\varphi|^2 - c \cdot \nabla\varphi + g\eta = -\sigma \nabla \cdot N \quad \text{on } S,$$

$$(2.1d) \quad \eta \rightarrow 0 \quad \text{as } |x'| \rightarrow \infty,$$

$$(2.1e) \quad \varphi, \nabla\varphi \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where here $g > 0$ is the constant acceleration due to gravity, $\sigma \geq 0$ is the constant coefficient of surface tension, $c = (c', 0) \in \mathbb{R}^n$ is the (nonzero) wave speed, and $N = N(x')$ is the unit normal vector to S pointing out of Ω .

Our main assumptions are that φ satisfies

$$(2.2a) \quad \varphi = o\left(\frac{1}{|x|^{n-2}}\right) \quad \text{as } |x| \rightarrow \infty,$$

while η and its derivatives satisfy

$$(2.2b) \quad \eta = O\left(\frac{1}{|x'|^{n-1+\varepsilon}}\right), \quad \frac{\partial \eta}{\partial x'_i} = O\left(\frac{1}{|x'|^{n+\varepsilon}}\right),$$

$$\frac{\partial^2 \eta}{\partial x'_i \partial x'_j} = O\left(\frac{1}{|x'|^{n+1+\varepsilon}}\right),$$

as $|x'| \rightarrow \infty$ for some $\varepsilon \in (0, 1)$ and all i, j . Note that (2.2a) follows from (2.1e) when $n = 2$.

Our first result is that φ behaves like a dipole at infinity.

THEOREM 2.1. *Let $\eta \in C^2(\mathbb{R}^{n-1})$ and $\varphi \in C^2(\overline{\Omega})$ solve (2.1) with $\sigma \geq 0$ and suppose that the decay estimates (2.2) hold. Then there exists a “dipole moment” $\mu = (\mu', 0) \in \mathbb{R}^n$ such that φ satisfies*

$$(2.3) \quad \varphi = \frac{\mu \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n-1+\varepsilon}}\right),$$

$$\nabla \varphi = \nabla \frac{\mu \cdot x}{|x|^n} + O\left(\frac{1}{|x|^{n+\varepsilon}}\right), \quad \text{as } |x| \rightarrow \infty,$$

while η satisfies

$$(2.4) \quad \eta = \frac{1}{g|x'|^n} \left(c \cdot \mu - n \frac{(c \cdot x')(\mu \cdot x')}{|x'|^2} \right) + O\left(\frac{1}{|x'|^{n+\varepsilon}}\right) \quad \text{as } |x'| \rightarrow \infty.$$

Dipole asymptotics along the lines of (2.3) feature in Longuet-Higgins’s numerical calculations of two-dimensional gravity-capillary waves [31, sec. 6] as well as Benjamin and Olver’s discussion of conserved quantities in the two-dimensional time-dependent problem in [6, sec. 6.5]. Compared to small-amplitude expansions, the asymptotic formula (2.4) is consistent with equation (4.1) in [2] in two dimensions and equation (5.6) in [27] in three dimensions when μ and c are parallel.

For two-dimensional gravity-capillary waves, Sun proves, roughly, that the decay $\eta = O(|x'|^{-1-\varepsilon})$ forces $\eta = O(|x'|^{-2})$ [36]. Our result is stronger in that it identifies the leading-order term in the asymptotics. On the other hand, Sun also allows for a semi-infinite upper layer with a different density, and, in the important special case when $\sigma > |c|^2/4g$, only needs to assume $\eta = O(|x'|^{-\varepsilon})$. As mentioned at the end of Section 1, his proof uses conformal mappings and hence does not generalize to three dimensions.

The decay rates in (2.2) also match those in Hur’s nonexistence result [22] for two-dimensional gravity waves. While weakening these decay assumptions is of course desirable, we have been unable to do so; see Appendix B.3 for a discussion of the difficulties involved. We can make a slight improvement in Corollary 2.4; see the remark in Section 4.

Our next result is an integral identity involving the dipole moment μ from Theorem 2.1.

THEOREM 2.2. *In the setting of Theorem 2.1, the kinetic energy, dipole moment μ , and wave speed c are related by*

$$(2.5) \quad \frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 dx = -\frac{\pi^{n/2}}{2\Gamma(\frac{n}{2})} (c \cdot \mu).$$

For two-dimensional gravity-capillary waves, Longuet-Higgins [31] gave a formal proof of (2.5) assuming (2.3); also see equation (6.22) in [6]. For three-dimensional waves, however, (2.5) seems to be new.

Theorem 2.2 implies $c \cdot \mu < 0$ for nontrivial waves with $\varphi \not\equiv 0$, and hence that the leading-order terms in (2.3) and (2.4) do not vanish. For $n = 2$, solving (2.5) for μ and substituting into (2.4) yields

$$\eta = \left(\frac{\pi}{g} \int_{\Omega} |\nabla \varphi|^2 dx \right) \frac{1}{|x'|^2} + O\left(\frac{1}{|x'|^{2+\varepsilon}} \right),$$

which, for instance, implies that $\eta > 0$ for $|x'|$ sufficiently large. For $n = 3$, we instead find that η takes both positive and negative values in any neighborhood of infinity. This in particular rules out waves of elevation (with $\eta > 0$) and waves of depression (with $\eta < 0$). For three-dimensional gravity waves, Craig [13] has already ruled out waves of elevation and depression using maximum principle arguments, without imposing any assumptions on the decay rates of η and φ . For three-dimensional gravity-capillary waves, on the other hand, these maximum principle arguments break down and the nonexistence of waves of elevation or depression seems to be new.

Another consequence of Theorem 2.2 is the following dichotomy:

COROLLARY 2.3. *In the setting of Theorem 2.1, either the angular momentum $\int_{\Omega} x \times \nabla \varphi dx$ is infinite or the wave is trivial, i.e., $\varphi \equiv 0$ and $\eta \equiv 0$.*

For two-dimensional time-dependent waves, Benjamin and Olver observe that $\mu' \neq 0$ causes the integral defining the total angular momentum to diverge [6, sec. 6.5]. This motivates them to impose additional restrictions guaranteeing that $\mu' = 0$, but not necessarily $\mu_n = 0$. Longuet-Higgins [31] explains for two-dimensional gravity-capillary waves how dipole behavior for the velocity potential causes the horizontal momentum to be indeterminate in that, for instance, $\nabla \varphi \notin L^1(\Omega)$.

A final corollary of the proof of Theorem 2.2 is that the excess mass vanishes.

COROLLARY 2.4. *In the setting of Theorem 2.1, the wave has zero excess mass in that*

$$(2.6) \quad \int_{\mathbb{R}^{n-1}} \eta(x') dx' = 0.$$

An obvious consequence is that no nontrivial waves satisfying (2.2) are waves of elevation or depression. For two-dimensional capillary-gravity waves, (2.6) was derived in [31], and a stronger version of Corollary 2.4 was proved rigorously in

[36]. For three-dimensional waves, however, Corollary 2.4 appears to be new. The nonexistence of three-dimensional waves of elevation and depression is new only in the capillary-gravity case; see the discussion after the statement of Theorem 2.2.

3 Proof of Theorem 2.1

The main ingredient in the proof of Theorem 2.1 is the following lemma, which states that (2.3) holds independently of the dynamic boundary condition (2.1c).

LEMMA 3.1. *Let $\varphi \in C^2(\overline{\Omega})$ and $\eta \in C^2(\mathbb{R}^{n-1})$ solve (2.1a)–(2.1b). If the decay estimates (2.2) hold, then there exists $\mu = (\mu', 0) \in \mathbb{R}^n$ (possibly zero) so that φ satisfies the asymptotic conditions (2.3).*

PROOF. We apply the Kelvin transform, setting

$$\tilde{x} = T(x) = \frac{x}{|x|^2}, \quad \tilde{\varphi}(\tilde{x}) = \frac{1}{|\tilde{x}|^{n-2}} \varphi\left(\frac{\tilde{x}}{|\tilde{x}|^2}\right), \quad \Omega^\sim = T(\Omega \setminus B_1),$$

where $B_1 = \{x : |x| < 1\}$ is an open ball centered at the origin. Note that $T(T(x)) = x$. This change of variables converts asymptotic questions about φ as $|x| \rightarrow \infty$ into local questions about $\tilde{\varphi}$ in a neighborhood of $0 \in \partial\Omega^\sim$. For instance, (2.2a) implies that $\tilde{\varphi}$ extends to a $C^0(\overline{\Omega^\sim})$ function with $\tilde{\varphi}(0) = 0$.

Using the decay assumptions (2.2), we show in the appendix that $\Omega^\sim$ has a C^2 boundary portion $S^\sim$ containing 0 and that $\tilde{\varphi} \in H^1(\Omega^\sim)$ is a weak solution to the boundary value problem

$$(3.1) \quad \Delta \tilde{\varphi} = 0 \quad \text{in } \Omega^\sim, \quad \frac{\partial \tilde{\varphi}}{\partial \tilde{N}} + \alpha \tilde{\varphi} = \beta \quad \text{on } S^\sim,$$

where $\alpha, \beta \in C^\varepsilon(S^\sim)$ are given up to a sign by

$$\alpha(\tilde{x}) = -(n-2)(x \cdot N(x)), \quad \beta(\tilde{x}) = |x|^n(c \cdot N(x)), \quad \text{on } S^\sim \setminus \{0\},$$

and $\alpha(0) = \beta(0) = 0$. Standard elliptic regularity theory (for instance, theorem 5.51 in [30]) then implies that $\tilde{\varphi} \in C^{1+\varepsilon}(\Omega^\sim \cup S^\sim)$. In particular, setting $\mu = (\mu', \mu_n) = \nabla \tilde{\varphi}(0)$, we have an expansion

$$(3.2) \quad \tilde{\varphi}(\tilde{x}) = \mu \cdot \tilde{x} + O(|\tilde{x}|^{1+\varepsilon}), \quad \nabla \tilde{\varphi}(\tilde{x}) = \mu + O(|\tilde{x}|^\varepsilon)$$

as $\tilde{x} \rightarrow 0$. Rewriting (3.2) in terms of φ and $\nabla \varphi$ then yields (2.3) as desired. Finally, plugging $\tilde{x} = 0$ in the boundary condition in (3.1), we find

$$\mu_n = \frac{\partial \tilde{\varphi}}{\partial \tilde{N}}(0) = -\alpha(0)\tilde{\varphi}(0) + \beta(0) = 0. \quad \square$$

Theorem 2.1 now follows from Lemma 3.1 and the dynamic boundary condition (2.1c).

PROOF OF THEOREM 2.1. We have already shown (2.3), so it suffices to prove (2.4). From (2.2b), we have $\nabla \cdot N = O(1/|x'|^{n+1+\varepsilon})$. Solving the dynamic boundary condition (2.1c) for η and plugging in (2.3) therefore yields

$$(3.3) \quad \begin{aligned} \eta(x') &= \frac{1}{g|x|^n} \left(c \cdot \mu - n \frac{(c \cdot x')(\mu \cdot x')}{|x|^2} \right) \\ &\quad + O\left(\frac{1}{|x|^{n+\varepsilon}}\right) + O\left(\frac{1}{|x'|^{n+1+\varepsilon}}\right), \end{aligned}$$

where here x is shorthand for $(x', \eta(x'))$. Since $\eta \rightarrow 0$ as $|x'| \rightarrow \infty$, we can replace each occurrence of x in (3.3) with $(x', 0)$, yielding (2.4) as desired. $\square$

4 Integral Identities

Let $B_r = \{x : |x| < r\}$ denote the open ball with radius r centered at the origin, and let $e_n = (0, 1)$ be the unit vector in the vertical direction.

PROOF OF THEOREM 2.2. Consider the vector field

$$\begin{aligned} A &:= \left(-\frac{|c|^2}{g}(e_n \cdot \nabla \varphi) + c \cdot x + \varphi \right) \nabla \varphi \\ &\quad + \frac{|c|^2}{g} \left(\frac{1}{2} |\nabla \varphi|^2 - c \cdot \nabla \varphi \right) e_n + \left(\frac{|c|^2}{g}(e_n \cdot \nabla \varphi) - \varphi \right) c. \end{aligned}$$

A simple calculation using only the fact that φ is harmonic shows that $\nabla \cdot A = |\nabla \varphi|^2$. Thus we can apply the divergence theorem to A on the bounded region $B_r \cap \Omega$ to obtain

$$(4.1) \quad \int_{B_r \cap \Omega} |\nabla \varphi|^2 dx = \int_{B_r \cap S} A \cdot N dS + \int_{\partial B_r \cap \Omega} A \cdot N dS.$$

Note that $B_r \cap S$ is the portion of the boundary of $B_r \cap \Omega$ on the free surface while $\partial B_r \cap \Omega$ is the portion inside the fluid.

On the free surface S , we have

$$(4.2) \quad N = \frac{(-\nabla \eta, 1)}{\sqrt{1 + |\nabla \eta|^2}}, \quad dS = \sqrt{1 + |\nabla \eta|^2} dx',$$

while the boundary conditions (2.1b) and (2.1c) imply

$$A \cdot N = (c \cdot x)(c \cdot N) - \left(\frac{|c|^2 \sigma}{g} \nabla \cdot N + |c|^2 \eta \right) (e_n \cdot N).$$

Thus the first term on the right-hand side of (4.1) can be rewritten as

$$\begin{aligned}
 \int_{B_r \cap S} A \cdot N \, dS &= - \int_{B_r \cap S} \left((c \cdot x)(c \cdot \nabla \eta) + \frac{|c|^2 \sigma}{g} \nabla \cdot N + |c|^2 \eta \right) dx' \\
 &= - \int_{B_r \cap S} \nabla \cdot \left(\frac{|c|^2 \sigma}{g} N + \eta(c \cdot x)c \right) dx' \\
 (4.3) \qquad &= \int_{\partial B_r \cap S} \left(\frac{|c|^2 \sigma}{g} N + \eta(c \cdot x)c \right) \cdot \nu' \, ds,
 \end{aligned}$$

where here the outward-pointing normal $\nu': T \rightarrow \mathbb{R}^{n-1}$ and measure ds are with respect to the projection of $\partial B_r \cap S$ onto $\mathbb{R}^{n-1}$.

Plugging (4.3) into (4.1) we obtain

$$\begin{aligned}
 (4.4) \quad \int_{\Omega \cap B_r} |\nabla \varphi|^2 \, dx &= \frac{|c|^2 \sigma}{g} \int_{S \cap \partial B_r} N \cdot \nu' \, ds - \int_{S \cap \partial B_r} \eta(c \cdot x)(c \cdot \nu') \, ds \\
 &\quad + \int_{\Omega \cap \partial B_r} A \cdot N \, dS.
 \end{aligned}$$

From (4.2) and (2.2b) we see that the first integrand on the right-hand side of (4.4) is $O(|\nabla \eta|) = O(|x'|^{-(n+\varepsilon)})$, while the second integrand is $O(|x'|^{-(n-2+\varepsilon)})$. Thus these first two integrals vanish as $r \rightarrow \infty$. Thanks to the asymptotic conditions (2.3) proved in Theorem 2.1, the remaining integral converges, as $r \rightarrow \infty$, to the constant value

$$\begin{aligned}
 &\int_{\partial B_r \cap \{x_n < 0\}} \left((c \cdot x) \nabla \frac{\mu \cdot x}{|x|^n} - \frac{\mu \cdot x}{|x|^n} c \right) \cdot \frac{x}{|x|} \, dS \\
 &= -n \int_{\partial B_1 \cap \{x_n < 0\}} (c \cdot x)(\mu \cdot x) \, dS = -\frac{\pi^{n/2}}{\Gamma(\frac{n}{2})} (c \cdot \mu),
 \end{aligned}$$

leaving us with (2.5) as desired. $\square$

PROOF OF COROLLARY 2.4. We follow the proof of Theorem 2.2, but with A replaced by the vector field

$$\tilde{A} = -\frac{|c|^2}{g} (e_n \cdot \nabla \varphi) \nabla \varphi + \frac{|c|^2}{g} \left(\frac{1}{2} |\nabla \varphi|^2 - c \cdot \nabla \varphi \right) e_n + \frac{|c|^2}{g} (e_n \cdot \nabla \varphi) c$$

obtained by dropping all of the terms in A without a factor of $|c|^2/g$. A simple calculation shows that $\nabla \cdot \tilde{A} = 0$, and the boundary conditions (2.1b) and (2.1c) give

$$\tilde{A} \cdot N = -|c|^2 \eta - \frac{|c|^2 \sigma}{g} \nabla \cdot N$$

on the free surface S . As in the proof of Theorem 2.2, we apply the divergence theorem, first to $\tilde{A}$ in $B_r \cap \Omega$, and then again on $S \cap B_r$, obtaining

$$(4.5) \quad |c|^2 \int_{B_r \cap S} \eta \, dx' = -\frac{|c|^2 \sigma}{g} \int_{S \cap \partial B_r} N \cdot v' \, ds + \int_{\Omega \cap \partial B_r} \tilde{A} \cdot N \, dS.$$

The first term on the right-hand side of (4.5) vanishes as $r \rightarrow \infty$ as in the proof of Theorem 2.2. The second term vanishes since $\tilde{A} = O(|\nabla \varphi|) = O(|x|^{-n})$ by (2.3). From (2.2b) we know that $\eta \in L^1(\mathbb{R}^{n-1})$, so taking $r \rightarrow \infty$ in (4.5) yields (2.6) as desired. $\square$

Remark 4.1. Unlike in the proofs of Theorem 2.2 and Corollary 2.3, the above proof of Corollary 2.4 does not make full use of the dipole asymptotics (2.3). Indeed, the same argument can be applied to any solution $\eta \in C^2(\mathbb{R}^{n-1})$, $\varphi \in C^2(\bar{\Omega})$ of (2.1) with $\sigma = 0$ and $\nabla \varphi = o(1/|x|^{n-1})$ to show that

$$(4.6) \quad \int_{B_r \cap S} \eta(x') \, dx' \rightarrow 0 \quad \text{as } r \rightarrow \infty$$

(it is no longer guaranteed that $\eta \in L^1$). The same is true with $\sigma > 0$ under the additional assumption that $\nabla \eta = o(1/|x'|^{n-2})$. Note that the weaker conclusion (4.6) still rules out waves of elevation and depression.

PROOF OF COROLLARY 2.3. For any $r > 0$, the asymptotic condition (2.3) implies that the integral

$$\int_{\Omega \cap \partial B_r} x \times \nabla \varphi \, dS$$

converges, as $r \rightarrow \infty$, to the constant value

$$(4.7) \quad \begin{aligned} \int_{\partial B_r \cap \{x_n < 0\}} x \times \nabla \frac{\mu \cdot x}{|x|^n} \, dS &= -\mu \times \int_{\partial B_1 \cap \{x_n < 0\}} x \, dS \\ &= \frac{\pi^{(n-1)/2}}{\Gamma(\frac{n+1}{2})} \mu \times e_n. \end{aligned}$$

Suppose that the angular momentum is finite. Then the right-hand side of (4.7) must be 0, which forces $\mu \times e_n = 0$ and hence $\mu' = 0$. But then $c \cdot \mu = 0$ so that (2.5) gives $\nabla \varphi \equiv 0$ and therefore $\varphi \equiv 0$.

It remains to show $\eta \equiv 0$. Plugging $\varphi \equiv 0$ into (2.1c), we find $g\eta = -\sigma \nabla \cdot N$. At a positive maximum of η , this reduces to $0 < g\eta = \sigma \Delta \eta \leq 0$, which is a contradiction. Similarly, at a negative minimum of η we have $0 > g\eta = \sigma \Delta \eta \geq 0$, which is again a contradiction, and we conclude that $\eta \equiv 0$. $\square$

Appendix A Proof of Lemma 3.1

In this appendix we provide the remaining details in the proof of Lemma 3.1.

Setting $S^\sim = (B_\delta \cap T(S)) \cup \{0\} \subset \partial\Omega^\sim$ for δ sufficiently small, we first claim that $S^\sim$ is a C^2 graph $\tilde{x}_n = f(\tilde{x}')$, which will imply that $S^\sim$ is a C^2 boundary portion. As an intermediate step, we define yet another variable

$$x^* = \frac{x'}{|x'|^2},$$

which, on $S^\sim \setminus \{0\}$, is related to $\tilde{x}'$ via

$$(A.1) \quad \tilde{x}' = \frac{x^*}{1 + |x^*|^2 \eta^2(x^*/|x^*|^2)}.$$

Our decay assumptions (2.2b) easily imply that $|x^*|^2 \eta^2(x^*/|x^*|^2)$ extends to a C^2 function of x^* in a neighborhood of $x^* = 0$, which vanishes at $x^* = 0$ together with its first and second derivatives. Thus we can use the implicit function theorem to solve (A.1) for x^* as a C^2 function of $\tilde{x}'$. Expressing $\tilde{x}_n$ in terms of x^* ,

$$(A.2) \quad \tilde{x}_n = \frac{|x^*|^2 \eta(x^*/|x^*|^2)}{1 + |x^*|^2 \eta^2(x^*/|x^*|^2)};$$

(2.2b) similarly guarantees that $\tilde{x}_n$ can be extended to a C^2 function of x^* in a neighborhood of $x^* = 0$. Composing the C^2 mappings $x^* \mapsto \tilde{x}_n$ and $\tilde{x}' \mapsto x^*$ yields the desired equation $\tilde{x}_n = f(\tilde{x}')$ for $S^\sim$.

We now consider the boundary condition satisfied by $\tilde{\varphi}$ on $S^\sim$. A calculation shows that a C^1 unit normal $\tilde{N}$ on $S^\sim$ is given in terms of the normal vector N on S via the formula

$$\tilde{N}(\tilde{x}) = N(x) - 2 \frac{N(x) \cdot x}{|x|^2}.$$

For simplicity assume that $\tilde{N}$ points out of $\Omega^\sim$; otherwise the definitions of β and α below are off by an unimportant sign. Differentiating the identity

$$\varphi(x) = \frac{1}{|x|^{n-2}} \tilde{\varphi}\left(\frac{x}{|x|^2}\right)$$

and using the boundary condition (2.1b), we find that, on $S^\sim \setminus \{0\}$,

$$c \cdot N = \frac{\partial \varphi}{\partial N} = -(n-2) \frac{x \cdot N}{|x|^n} \tilde{\varphi} + \frac{1}{|x|^n} \frac{\partial \tilde{\varphi}}{\partial \tilde{N}}.$$

Multiplying through by $|x|^n$, we write this as

$$\frac{\partial \tilde{\varphi}}{\partial \tilde{N}} + \alpha \tilde{\varphi} = \beta \quad \text{on } S^\sim \setminus \{0\},$$

where α and β are defined on $S^\sim \setminus \{0\}$ by

$$\alpha(\tilde{x}) = -(n-2)(x \cdot N), \quad \beta(\tilde{x}) = |x|^n(c \cdot N).$$

Clearly $\alpha, \beta \in C^1(S^\sim \setminus \{0\})$. Using our decay assumptions (2.2b), we check that they extend to C^ε functions of $\tilde{x}'$ vanishing at $\tilde{x}' = 0$. We remark that only this last

extension of β requires the full force of (2.2b); the other extensions only require $\eta = O(1/|x|^\varepsilon)$, $D\eta = O(1/|x|^{1+\varepsilon})$, and $D^2\eta = O(1/|x|^{2+\varepsilon})$.

Next we show that $\tilde{\varphi} \in H^1(\Omega^\sim)$. From (2.2b) we know $\tilde{\varphi} \in C^0(\overline{\Omega^\sim}) \cap C^2(\Omega^\sim \setminus \{0\})$, so it is enough to show $\nabla \tilde{\varphi} \in L^2(\Omega^\sim \cap B_R)$ for some $R > 0$. Fix R small enough that $B_{2R} \cap \partial\Omega^\sim \subset S^\sim$, let $\chi_0 \in C_c^\infty(\mathbb{R})$ be a nonnegative function satisfying $\chi_0(s) = 0$ for $s < 1$ and $\chi_0(s) = 1$ for $s > 2$, and for $r < R/2$ define

$$\chi(\tilde{x}; r) = \chi_0(r^{-1}|\tilde{x}|)(1 - \chi_0(R^{-1}|\tilde{x}|)).$$

Multiplying $\Delta \tilde{\varphi} = 0$ by $\chi^2 \tilde{\varphi}$ and integrating by parts, we find

$$\begin{aligned} \int_{\Omega^\sim} |\chi \nabla \tilde{\varphi}|^2 d\tilde{x} &= -2 \int_{\Omega^\sim} \chi \tilde{\varphi} \nabla \chi \cdot \nabla \tilde{\varphi} d\tilde{x} + \int_{S^\sim} \chi \tilde{\varphi} (\beta - \alpha \tilde{\varphi}) dS \\ (A.3) \quad &\leq C(1 + \|\nabla \chi\|_{L^2(\mathbb{R}^n)} \|\chi \nabla \tilde{\varphi}\|_{L^2(\Omega^\sim)}), \end{aligned}$$

where C depends on the L^∞ norms of $\tilde{\varphi}$, α , and β . Since

$$\|\nabla \chi\|_{L^2(\mathbb{R}^n)} \leq C(1 + r^{n-2}) \leq C,$$

(A.3) implies an upper bound on $\|\chi \nabla \tilde{\varphi}\|_{L^2(\Omega^\sim)}$ independent of $r < R/2$. Sending $r \rightarrow 0$, we obtain $\nabla \tilde{\varphi} \in L^2(\Omega^\sim \cap B_R)$ as desired.

Finally, we claim that $\tilde{\varphi}$ is a weak solution to (3.1). Certainly (after perhaps changing the definitions of α , β by a sign)

$$(A.4) \quad \int_{\Omega^\sim} \nabla \tilde{\varphi} \cdot \nabla v d\tilde{x} = \int_{S^\sim} (\alpha \tilde{\varphi} - \beta) v d\tilde{x}$$

for all smooth $v \in H^1(\Omega^\sim)$ vanishing in a neighborhood of 0. Such v are dense in $H^1(\Omega^\sim)$ (see, for instance, lemmas 17.2 and 17.3 in [37]). Since $\tilde{\varphi} \in H^1(\Omega^\sim)$, (A.4) therefore holds for all $v \in H^1(\Omega^\sim)$ and the claim is proved.

Appendix B Forced Waves with Slow Decay

In this appendix we take $n = 2$ and $0 \leq \sigma < c^2/4g$ and consider forced solitary waves, i.e., solutions to (2.1) with an additional source term on the right-hand side of the dynamic boundary condition (2.1c) that enforces a nonconstant pressure on the free surface.

B.1 Formulation

To keep the presentation brief, we will be somewhat informal, and in order to simplify the calculations we switch to standard conformal variables. Identifying $(x', x_n) \in \mathbb{R}^2$ with $z = x' + ix_n \in \mathbb{C}$,

$$w(z) = \Phi(z) + i\Psi(z) = z - \frac{1}{c}(\varphi(z) + i\psi(z))$$

is a rescaled complex velocity potential in the moving frame. Here ψ is the harmonic conjugate of φ , which, thanks to the kinematic condition (2.1b), can be normalized so that $\psi = c\eta$ on the free surface. As is well-known, w maps the

fluid domain $y < \eta$ conformally onto the lower half-plane $\Psi < 0$. Writing the inverse of this mapping as

$$z = w + \zeta(w),$$

the asymptotic conditions (2.1d)–(2.1e) correspond to $\zeta, \zeta' \rightarrow 0$ as $|w| \rightarrow \infty$, while the dynamic boundary condition (2.1c) becomes

$$(B.1) \quad \operatorname{Re} \left\{ \frac{c^2}{2} \left(\frac{1}{|1 + \zeta'|^2} - 1 \right) - i g \zeta + \sigma c^2 \frac{i \zeta''}{(1 + \zeta')|1 + \zeta'|} \right\} = p \quad \text{on } \Psi = 0,$$

where here the right-hand side p represents pressure forcing. Since $\Phi \sim x$ as $|\Phi| \rightarrow \infty$, the decay rates of ζ and p in Φ coincide with their decay rates in x .

B.2 A Family of Explicit Examples

By our assumption $\sigma < c^2/4g$, there is a positive root k of the dispersion relation

$$(B.2) \quad -c^2 k^2 \sigma + c^2 k - g = 0,$$

corresponding to the linear periodic wave $\zeta_{\text{per}} = \varepsilon e^{ikw}$. Multiplying ζ_{per} by an algebraically decaying envelope, we set

$$(B.3) \quad \zeta = \varepsilon \frac{e^{ikw}}{(w - i)^q},$$

where $q \in (0, 1]$ is the decay rate. Certainly ζ is holomorphic in the lower half-plane and satisfies the asymptotic conditions $\zeta, \zeta' \rightarrow 0$ as $|w| \rightarrow \infty$. Choosing $\varepsilon > 0$ sufficiently small, there is no issue with the denominators on the left-hand side of (B.1), which we then view as the definition of the pressure forcing p . Unsurprisingly, the elevation of the free surface decays like $1/|x|^q$, with the asymptotic expansion

$$\eta = \operatorname{Im} \zeta(\Phi) = \varepsilon \frac{\sin(k\Phi + \delta_{\pm})}{|\Phi|^q} + O\left(\frac{1}{|\Phi|^{1+q}}\right)$$

as $\Phi \rightarrow \pm\infty$ for some phase shifts $\delta_{\pm}$ depending on q . By our choice of k , however, the pressure disturbance decays more quickly:

$$(B.4) \quad p = \varepsilon(-\sigma c^2 k^2 + c^2 k - g) \frac{\sin(k\Phi + \delta_{\pm})}{|\Phi|^q} + O\left(\frac{1}{|\Phi|^{2q}}\right) = O\left(\frac{1}{|\Phi|^{2q}}\right).$$

To first order in ε (i.e., for the linearized equations) the decay is still faster:

$$(B.5) \quad p_{\text{lin}} = O\left(\frac{1}{|\Phi|^{1+q}}\right).$$

B.3 Implications for a Class of Linear Arguments

A common method for improving the decay rates of solutions to nonlinear equations, used for instance in [14], can be roughly outlined as follows. First, the equation is written abstractly as

$$(B.6) \quad L\zeta = N(\zeta) + p$$

where L is a linear operator, N represents the nonlinear dependence on ζ , and p is the forcing term. Then a lemma is proved to the effect that an algebraic decay rate q of $p_{\text{lin}} = L\zeta$ is automatically inherited (or improved on) by ζ , provided q does not exceed some maximum rate q_{max} . Suppose first that the forcing term $p = 0$ and that ζ decays at a rate $q_1 > 0$. Then the nonlinear terms decay at twice this rate, and so by the lemma q_1 can be increased to $q_2 = \min(2q_1, q_{\text{max}})$. Continuing in this way we eventually find that ζ decays at the maximum admissible rate q_{max} . Clearly the same argument works with a nonzero forcing term p , provided it too decays at the maximum rate q_{max} .

For the forced water wave problem considered in this section, (B.4) shows that the above conclusion is *false*: ζ decays at the rate $q \in (0, 1]$ while the forcing p decays at the faster rate $2q$. Looking at (B.5), we see the reason: the lemma on the mapping properties of L does not hold. Indeed, the best replacement lemma that we can hope for is that a decay rate of $1 + q$ for $p_{\text{lin}} = L\zeta$ implies a rate of q for ζ . But again, such a lemma applied to (B.6) (say with $p = 0$) would only improve the decay rate q of ζ when $q < 2q - 1$, i.e., when $q > 1$.

This is why (at least for $\sigma < c^2/4g$) the arguments in [36] are limited to waves with decay rate $q > 1$, although we do note that the appendix of that paper contains some results on waves with a certain asymptotic expansion at infinity related to (B.3) with $q = 1$.

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Bibliography

- [1] Akers, B.; Milewski, P. A. A model equation for wavepacket solitary waves arising from capillary-gravity flows. *Stud. Appl. Math.* **122** (2009), no. 3, 249–274. doi:10.1111/j.1467-9590.2009.00432.x
- [2] Akylas, T. R.; Dias, F.; Grimshaw, R. H. J. The effect of the induced mean flow on solitary waves in deep water. *J. Fluid Mech.* **355** (1998), 317–328. doi:10.1017/S0022112097007714
- [3] Amick, C. J.; Kirchgässner, K. A theory of solitary water-waves in the presence of surface tension. *Arch. Rational Mech. Anal.* **105** (1989), no. 1, 1–49. doi:10.1007/BF00251596
- [4] Amick, C. J.; Toland, J. F. On solitary water-waves of finite amplitude. *Arch. Rational Mech. Anal.* **76** (1981), no. 1, 9–95. doi:10.1007/BF00250799
- [5] Beale, J. T. The existence of solitary water waves. *Comm. Pure Appl. Math.* **30** (1977), no. 4, 373–389. doi:10.1002/cpa.3160300402

- [6] Benjamin, T. B.; Olver, P. J. Hamiltonian structure, symmetries and conservation laws for water waves. *J. Fluid Mech.* **125** (1982), 137–185. doi:10.1017/S0022112082003292
- [7] Buffoni, B. Existence by minimisation of solitary water waves on an ocean of infinite depth. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21** (2004), no. 4, 503–516. doi:10.1016/j.anihpc.2003.06.003
- [8] Buffoni, B.; Groves, M. D. A multiplicity result for solitary gravity-capillary waves in deep water via critical-point theory. *Arch. Ration. Mech. Anal.* **146** (1999), no. 3, 183–220. doi:10.1007/s002050050141
- [9] Buffoni, B.; Groves, M. D.; Sun, S. M.; Wahlén, E. Existence and conditional energetic stability of three-dimensional fully localised solitary gravity-capillary water waves. *J. Differential Equations* **254** (2013), no. 3, 1006–1096. doi:10.1016/j.jde.2012.10.007
- [10] Buffoni, B.; Groves, M. D.; Toland, J. F. A plethora of solitary gravity-capillary water waves with nearly critical Bond and Froude numbers. *Philos. Trans. Roy. Soc. London Ser. A* **354** (1996), no. 1707, 575–607. doi:10.1098/rsta.1996.0020
- [11] Buffoni, B.; Groves, M. D.; Wahlén, E. A variational reduction and the existence of a fully localised solitary wave for the three-dimensional water-wave problem with weak surface tension. *Arch. Ration. Mech. Anal.* **228** (2018), no. 3, 773–820. doi:10.1007/s00205-017-1205-1
- [12] Chen, R. M.; Walsh, S.; Wheeler, M. H. Existence, nonexistence, and asymptotics of deep water solitary waves with localized vorticity. Preprint, 2017. arXiv:1706.00147 [math.AP]
- [13] Craig, W. Non-existence of solitary water waves in three dimensions. Recent developments in the mathematical theory of water waves (Oberwolfach, 2001). *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.* **360** (2002), no. 1799, 2127–2135. doi:10.1098/rsta.2002.1065
- [14] Craig, W.; Sternberg, P. Symmetry of solitary waves. *Comm. Partial Differential Equations* **13** (1988), no. 5, 603–633. doi:10.1080/03605308808820554
- [15] Deng, Y.; Ionescu, A. D.; Pausader, B.; Pusateri, F. Global solutions of the gravity-capillary water-wave system in three dimensions. *Acta Math.* **219** (2017), no. 2, 213–402. doi:10.4310/ACTA.2017.v219.n2.a1
- [16] Dias, F.; Kharif, C. Nonlinear gravity and capillary-gravity waves. *Annual Review of Fluid Mechanics* **31**, 301–346 (1999). doi:10.1146/annurev.fluid.31.1.301
- [17] Friedrichs, K. O.; Hyers, D. H. The existence of solitary waves. *Comm. Pure Appl. Math.* **7** (1954), 517–550. doi:10.1002/cpa.3160070305
- [18] Germain, P.; Masmoudi, N.; Shatah, J. Global solutions for the gravity water waves equation in dimension 3. *Ann. of Math. (2)* **175** (2012), no. 2, 691–754. doi:10.4007/annals.2012.175.2.6
- [19] Groves, M. D. Steady water waves. *J. Nonlinear Math. Phys.* **11** (2004), no. 4, 435–460. doi:10.2991/jnmp.2004.11.4.2
- [20] Groves, M. D.; Sun, S.-M. Fully localised solitary-wave solutions of the three-dimensional gravity-capillary water-wave problem. *Arch. Ration. Mech. Anal.* **188** (2008), no. 1, 1–91. doi:10.1007/s00205-007-0085-1
- [21] Groves, M. D.; Wahlén, E. On the existence and conditional energetic stability of solitary gravity-capillary surface waves on deep water. *J. Math. Fluid Mech.* **13** (2011), no. 4, 593–627. doi:10.1007/s00021-010-0034-x
- [22] Hur, V. M. No solitary waves exist on 2D deep water. *Nonlinearity* **25** (2012), no. 12, 3301–3312. doi:10.1088/0951-7715/25/12/3301
- [23] Hur, V. M. Kinetic, potential and surface tension energies of solitary waves in deep water. *J. Phys. A* **48** (2015), no. 42, 42FT01, 8 pp. doi:10.1088/1751-8113/48/42/42FT01
- [24] Iooss, G.; Kirchgässner, K. Water waves for small surface tension: an approach via normal form. *Proc. Roy. Soc. Edinburgh Sect. A* **122** (1992), no. 3–4, 267–299. doi:10.1017/S0308210500021119
- [25] Iooss, G.; Kirrmann, P. Capillary gravity waves on the free surface of an inviscid fluid of infinite depth. Existence of solitary waves. *Arch. Rational Mech. Anal.* **136** (1996), no. 1, 1–19. doi:10.1007/BF02199364

- [26] Keady, G.; Pritchard, W. G. Bounds for surface solitary waves. *Proc. Cambridge Philos. Soc.* **76** (1974), 345–358.
- [27] Kim, B.; Akylas, T. R. On gravity-capillary lumps. *J. Fluid Mech.* **540** (2005), 337–351. doi:10.1017/S0022112005005823
- [28] Kirchgässner, K. Nonlinearly resonant surface waves and homoclinic bifurcation. *Advances in applied mechanics, Vol. 26*, 135–181. Advances in Applied Mechanics, 26. Academic Press, Boston, 1988. doi:10.1016/S0065-2156(08)70288-6
- [29] Lavrent'ev, M. A. I. On the theory of long waves. II. A contribution to the theory of long waves. *Amer. Math. Soc. Translation* **1954** (1954), no. 102, 53 pp.
- [30] Lieberman, G. M. *Oblique derivative problems for elliptic equations*. World Scientific, Hackensack, N.J., 2013. doi:10.1142/8679
- [31] Longuet-Higgins, M. S. Capillary-gravity waves of solitary type on deep water. *J. Fluid Mech.* **200** (1989), 451–470. doi:10.1017/S002211208900073X
- [32] Mielke, A. Reduction of quasilinear elliptic equations in cylindrical domains with applications. *Math. Methods Appl. Sci.* **10** (1988), no. 1, 51–66. doi:10.1002/mma.1670100105
- [33] Miles, J. W. Solitary waves. *Annual review of fluid mechanics, Vol. 12*, 11–43. Annual Reviews, Palo Alto, 1980.
- [34] Părău, E. I.; Vanden-Broeck, J.-M.; Cooker, M. J. Nonlinear three-dimensional gravity-capillary solitary waves. *J. Fluid Mech.* **536** (2005), 99–105. doi:10.1017/S0022112005005136
- [35] Shatah, J.; Walsh, S.; Zeng, C. Travelling water waves with compactly supported vorticity. *Nonlinearity* **26** (2013), no. 6, 1529–1564. doi:10.1088/0951-7715/26/6/1529
- [36] Sun, S. M. Some analytical properties of capillary-gravity waves in two-fluid flows of infinite depth. *Proc. Roy. Soc. London Ser. A* **453** (1997), no. 1961, 1153–1175. doi:10.1098/rspa.1997.0064
- [37] Tartar, L. *An introduction to Sobolev spaces and interpolation spaces*. Lecture Notes of the Unione Matematica Italiana, 3. Springer, Berlin; UMI, Bologna, 2007.
- [38] Vanden-Broeck, J.-M.; Dias, F. Gravity-capillary solitary waves in water of infinite depth and related free-surface flows. *J. Fluid Mech.* **240** (1992), 549–557. doi:10.1017/S0022112092000193
- [39] Wang, X. Global solution for the 3D gravity water waves system above a flat bottom. Preprint, 2015. arXiv:1508.06227 [math.AP]
- [40] Wang, X. On the 3-dimensional water waves system above a flat bottom. *Anal. PDE* **10** (2017), no. 4, 893–928. doi:10.2140/apde.2017.10.893
- [41] Wang, Z.; Milewski, P. A. Dynamics of gravity-capillary solitary waves in deep water. *J. Fluid Mech.* **708** (2012), 480–501. doi:10.1017/jfm.2012.320
- [42] Wu, S. Global wellposedness of the 3-D full water wave problem. *Invent. Math.* **184** (2011), no. 1, 125–220. doi:10.1007/s00222-010-0288-1

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