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Asymptotic formulas for solitary waves in the high-energy limit of FPU-type chains

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Abstract. It is well established that the solitary waves of FPU-type chains converge in the high-energy limit to traveling waves of the hard-sphere model. In this paper we establish improved asymptotic expressions for the wave profiles as well as an explicit formula for the wave speed. The key step in our approach is the derivation of an asymptotic ODE for the appropriately rescaled strain profile.

AMS classification scheme numbers: 37K60, 37K40, 74H10

Keywords: asymptotic analysis, lattice waves, high-energy-limit of FPU-type chains

1. Introduction

Traveling waves in nonlinear Hamiltonian lattice systems are ubiquitous in many branches of sciences and their mathematical analysis has attracted a lot of interest over the last two decades. In the simplest case of a spatially one-dimensional lattice with nearest-neighbor interactions – often called Fermi-Pasta-Ulam or FPU-type chain – the analytical problem consists of finding a positive wave-speed parameter σ along with a distance profile R and a velocity profile V such that

$$\begin{aligned} R'(x) \quad &= V\big(x+\tfrac{1}{2}\big) - V\big(x-\tfrac{1}{2}\big)\,, \\ \sigma\, V'(x) &= \Phi'\Big(R\big(x+\tfrac{1}{2}\big)\Big) - \Phi'\Big(R\big(x-\tfrac{1}{2}\big)\Big) \end{aligned} \tag{1}$$

is satisfied for all $x \in \mathbb{R}$. Here Φ is the nonlinear interaction potential and the position $u_j(t)$ of particle j can be recovered by

$$u_j(t) = U\big(j - \sqrt{\sigma}\, t\big), \qquad U(x) := \int_{x_0}^{x} V(y)\,\mathrm{d}y\,,$$

which implies the identities

$$\dot{u}_j(t) = \sqrt{\sigma}\, V\big(j - \sqrt{\sigma}\, t\big) \quad \text{and} \quad u_{j+1}(t) - u_j(t) = R\big(j + \tfrac{1}{2} - \sqrt{\sigma}\, t\big)$$

for the atomic velocities and distances, respectively. In particular, u satisfies Newton's law of motion

$$\ddot{u}_j(t) = \Phi'(u_{j+1}(t) - u_j(t)) - \Phi'(u_j(t) - u_{j-1}(t))\,, \qquad j \in \mathbb{Z}\,. \tag{2}$$

The existence of several types of traveling wave solutions (with periodic, solitary, front-like, or even more complex profile functions) can be established in different frameworks; see, for instance, [5, 2] for constrained optimization problems, [9] for critical point techniques, [8] for spatial dynamics, and [10, 13] for almost explicit solutions. However, very little is known about the uniqueness of the solutions to the advance-delay differential equation (1) or their dynamical stability within (2). The only nonlinear exceptions are the Toda chain – which is completely integrable, see [11] and references therein – and the Korteweg-de Vries (KdV) limit of solitary waves in chains with so called hardening. The latter has been investigated by Friesecke and Pego in a series of four seminal papers starting with [4]. In this limit, solitary waves have small amplitudes, carry low energy, and are spread over a huge number of lattice sites. The discrete difference operators in (1) can therefore be approximated by continuous differential operators and the asymptotic properties are governed by the KdV equation, which is completely integrable and well understood.

Another interesting asymptotic regime concerns solitary waves with high energy in chains with rapidly increasing potential. Here the profile functions localize completely since V converges – maybe after some affine rescaling – to the indicator function of an interval, see [3, 12] or [6] for potentials Φ that posses a singularity or grow super-polynomially, respectively. The physical interpretation of the high-energy limit is that the particles interact asymptotically as in the hard-sphere limit, that means by elastic collisions only.

The high-energy limit is another natural candidate for tackling the analytical problems concerning the uniqueness and the stability of traveling wave. In this context we are especially interested in the spectral properties of the linearized traveling waves equation – see the discussion in section 3.6 – but the convergence results from the aforementioned papers do not give any control in this direction. They are too weak and provide neither an explicit leading order formula for σ nor the next-to-leading order corrections to the asymptotic profile functions. In this paper we derive such formulas and present a refined asymptotic analysis of the high-energy limit for potentials with sufficiently strong singularity.

1.1. The high-energy limit

In order to keep the presentation as simple as possible, we restrict our considerations to the example potential

$$\Phi(r) = \frac{1}{m(m+1)}\left(\frac{1}{(1-r)^m} - m\,r - 1\right) \qquad \text{with} \quad m \in (1,\,\infty)\,, \tag{3}$$

which is convex and well-defined for $r < 1$, satisfies

$$\Phi(0) = \Phi'(0) = 0\,, \qquad \Phi''(0) = 1\,,$$

and becomes singular as $r \nearrow 1$. The condition $m > 1$ is quite essential and shows up several times in our proofs. The other details are less important and our asymptotic approach can hence be generalized to the case

$$\begin{aligned}&\Phi \text{ is convex and smooth on some interval } [a,\,b] \text{ with } \Phi'(a) = 0\\ &\text{such that the limit } \lim_{x\nearrow b}\Phi(x)(b-x)^m \text{ does exist}\,.\end{aligned}$$

This class also includes – after a reflection with respect to the distance variable – all Lennard-Jones-type potential, which blow up on the left of the global minimum.

To simplify the exposition further, we merely postulate the existence of a family of solitary waves with certain properties but sketch in section 1.4 how our assumption can be justified rigorously. Specifically, we rely on the following standing assumption, where *unimodal profile* means increasing and decreasing for negative and positive x, respectively.

Assumption 1 (family of high-energy waves)**.** $(V_\delta,\,R_\delta,\,\sigma_\delta)_{0<\delta<1}$ *is a family of solitary waves with the following properties:*

(i) V_δ *and* R_δ *belong to* $\mathsf{L}^2(\mathbb{R}) \cap \mathsf{BC}^1(\mathbb{R})$ *and are nonnegative, even, and unimodal.*

(ii) V_δ *is normalized by* $\|V_\delta\|_2 = 1-\delta$ *and* R_δ *takes values in* $[0,\,1)$.

Moreover, the potential energy explodes in the sense of $p_\delta := \int_{\mathbb{R}} \Phi(R_\delta(x))\,\mathrm{d}x \to +\infty$ *as* $\delta \to 0$.

Beside of δ there exist two other small quantities, namely

$$\varepsilon_\delta := 1 - R_\delta(0)\,, \qquad \mu_\delta := \sqrt{\sigma_\delta\,\varepsilon_\delta^{m+2}}, \tag{4}$$

which feature prominently in the asymptotic analysis. The amplitude parameter ε_δ quantifies the impact of the singularity and appears naturally in many of the estimates derived below. The parameter μ_δ, which looks rather artificial at a first glance, is also very important as it determines the length scale for the leading order corrections to the asymptotic profile functions V_0 and R_0.

For the interaction potential (3) and the waves from Assumption 1, the existing results for the limit $\delta \to 0$ are illustrated in Figures 1, 2 and can be summarized as follows.

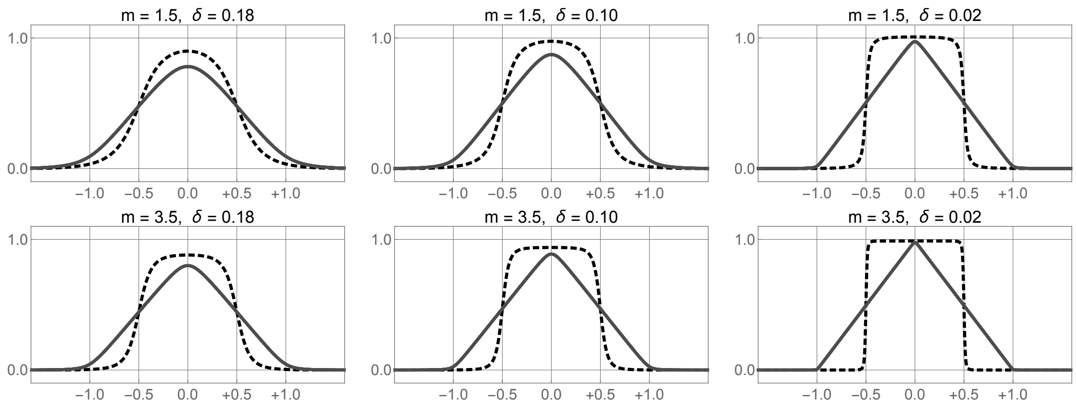

Figure 1. Numerical examples of solitary waves for the potential (3) and as in Assumption 1: The graphs of V_δ (black, dashed) and R_δ (gray, solid) are plotted for $m = 1.5$ (top row) and $m = 2.5$ (bottom row). In the high-energy limit $\delta \to 0$ (from left to right column), V_δ and R_δ approach the indicator function V_0 and the tent map R_0, respectively. See section 1.4 for details concerning the numerical scheme.

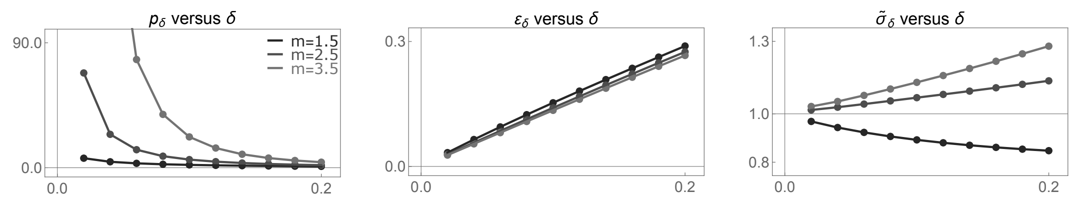

Figure 2. Parameter plots for three different choices of m and the simulations from Figure 1: p_δ represents the potential energy, ε_δ is the amplitude parameter from (4), and $\tilde{\sigma}_\delta := \sigma_\delta / \left(\overline{\mu}^2 \varepsilon_\delta^m\right)$ measures the relative deviation of the speed parameter σ with respect to the asymptotic value from (32)

Theorem 2 (localization theorem)**.** *In the high-energy limit, we have*

$$\|V_\delta - V_0\|_2 + \|R_\delta - R_0\|_\infty + \varepsilon_\delta + \mu_\delta \quad \xrightarrow{\delta \to 0} \quad 0$$

with

$$V_0(x) := \chi(x) \qquad R_0(x) := \max\{0,\, 1 - |x|\,\}\,,$$

where χ denotes the indicator function of the interval $[-\tfrac{1}{2},\, +\tfrac{1}{2}]$.

We give a short proof in section 1.3. The corresponding results in [3, 12] also provide lower and upper bounds but no explicit expansion for σ_δ.

1.2. Statement of the asymptotic result

Our strategy for deriving a refined asymptotic analysis is to blow up the profile functions near the critical spatial positions and to identify equations that determine the asymptotic wave shape with respect to a rescaled space variable $\tilde{x}$. Specifically, we use the *transition scaling* in order to describe the asymptotic velocity profile near $x = \pm\frac{1}{2}$, while the distance profile can be rescaled at both the *tip position* $x = 0$ and the

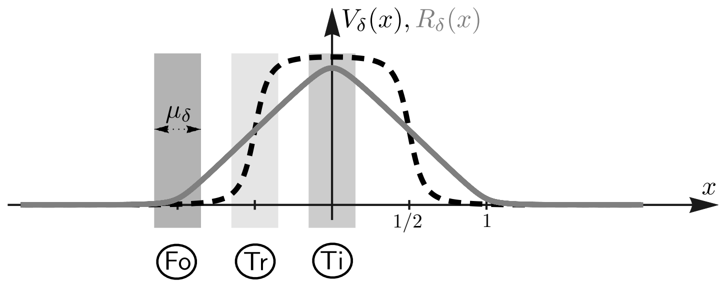

Figure 3. Schematic representation of the different scalings: The transition scaling describes the jump-like behavior of V_δ near $x = \pm\frac{1}{2}$ while the foot and the tip scaling magnify the turns of R_δ at $x \approx \pm 1$ and $x \approx 0$, respectively. The width parameter μ_δ is introduced in (4) and satisfies $\mu_\delta \sim \varepsilon_\delta$ according to Corollary 13.

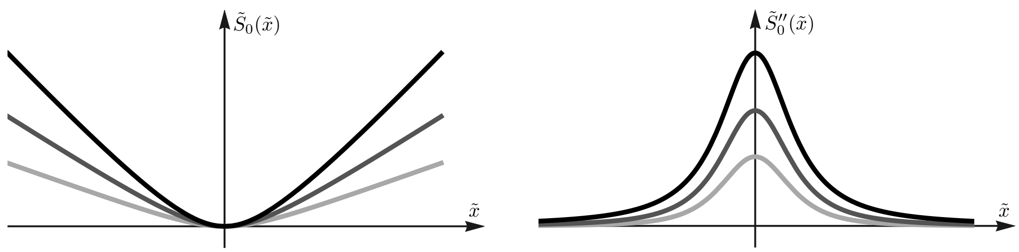

Figure 4. Graph of the function $\tilde{S}_0$ and its second derivative for $m = m_1$ (black) and $m = m_2$ (dark gray) and $m = m_3$ (light gray) with $m_1 < m_2 < m_3$.

foot positions $x = \pm 1$, see Figure 3 for an illustration. Our main findings can informally be summarized as follows.

Main results. *In the high-energy limit* $\delta \to 0$, *all relevant information on* $(R_\delta, V_\delta, \sigma_\delta)$ *can be obtained from the function* $\tilde{S}_0$, *which is defined by the ODE initial value problem*

$$\tilde{S}_0''(\tilde{x}) = \frac{2}{m+1} \cdot \frac{1}{\Big(1 + \tilde{S}_0(\tilde{x})\Big)^{m+1}}, \qquad \tilde{S}_0(0) = \tilde{S}_0'(0) = 0 \tag{5}$$

and plotted in Figure 4. More precisely,

(i) the velocity profile V_δ *converges under the* transition scaling,

(ii) the distance profile R_δ *converges under both the* tip scaling *and the* foot scaling,

(iii) the rescaled parameters $\delta^m \sigma_\delta$, $\delta^{-1}\varepsilon_\delta$, *and* $\delta^{-1}\mu_\delta$ *converge,*

where the respective limit objects can be expressed in terms of $\tilde{S}_0$ *and all error terms are at most of order* $O(\delta^m)$.

The details concerning the convergence under the tip, the transition, and the foot scaling are presented in the Theorems 7, 9, and 15, respectively, while Corollary 13 provides the explicit scaling laws for ε_δ, μ_δ, and σ_δ. Moreover, the combination of all partial estimates gives rise to the global approximation results in Theorem 16 and Corollary 17.

The above results provide an improved understanding of the high-energy limit of solitary waves. In particular, it seems that our asymptotic formulas can be used to control the spectrum of the linearized traveling wave equation, see the brief and preliminary discussion in section 3.6.

The paper is organized as follows. In the remainder of the introduction we prove the localization theorem and discuss both the results from [3, 12] and the justification of Assumption 1 in greater detail. Then section 2 is devoted to the tip scaling, which turns out to be most fundamental step in our asymptotic analysis. In particular, we identify the intrinsic scaling parameters in section 2.1 and link afterwards in section 2.2 the rescaled distance profile to the initial value problem (5). In section 3 we finally employ the results on the tip scaling and establish all other asymptotic formulas.

1.3. Preliminaries

In this section we prove Theorem 2 since it provides the starting point for our asymptotic analysis in section 2 and section 3. To this end it is convenient to reformulate the advance-delay-differential equation (1) as

$$R = AV\,, \qquad \sigma\, V = A\Phi'(R)\,, \tag{6}$$

where the operator A stands for the convolution with the indicator function χ. This reads

$$(AV)(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} V(y)\,\mathrm{d}y$$

and the elimination of R reveals that (1) can be viewed as a symmetric but nonlinear and nonlocal eigenvalue problem for the eigenvalue σ and the eigenfunction V. The proof that (6) implies (1) is straight forward and involves only differentiation with respect to x; for the reversed statement one has to eliminate the constants of integration by the decay condition $V \in \mathsf{L}^2(\mathbb{R})$.

Using elementary analysis such as Hölder's inequality we readily verify the estimates

$$\|AV\|_2 \le \|V\|_2\,, \qquad \|AV\|_\infty \le \|V\|_2\,, \qquad \|(AV)'\|_2 \le 2\,\|V\|_2\,, \tag{7}$$

for any function $V \in \mathsf{L}^2(\mathbb{R})$, and this implies that the potential energy

$$\mathcal{P}(V) := \int_{\mathbb{R}} \Phi((AV)(x))\,\mathrm{d}x\,, \tag{8}$$

is well defined as long as $\|V\|_2 < 1$. Moreover, we get

$$\|R_\delta\|_\infty = R_\delta(0) = 1 - \varepsilon_\delta \le \|V_\delta\|_2 = 1 - \delta \tag{9}$$

for the family from Assumption 1.

Lemma 3 (variant of the localization theorem)**.** *The estimates*

$$\|V_\delta - \chi\|_2 \le C\varepsilon_\delta, \qquad \|R_\delta - A\chi\|_\infty \le C\varepsilon_\delta \tag{10}$$

and

$$c\,\varepsilon_\delta \le \mu_\delta \le C\sqrt{\varepsilon_\delta} \tag{11}$$

hold for some constants c, C independent of δ. Moreover, we have $\varepsilon_\delta \to 0$ as $\delta \to 0$.

Proof. We start with the identities

$$R_\delta(0) = \langle V_\delta,\, \chi\rangle\,, \qquad \|V_\delta - \chi\|_2^2 = \|V_\delta\|_2^2 + \|\chi\|_2^2 - 2\langle V_\delta,\, \chi\rangle\,, \tag{12}$$

where $\langle\cdot,\,\cdot\rangle$ denotes the usual inner product in $\mathsf{L}^2(\mathbb{R})$, and observe that (6) implies

$$\sigma_\delta(1-\delta)^2 = \langle\Phi'(R_\delta),\, R_\delta\rangle\,. \tag{13}$$

Since (9) yields $\delta \le \varepsilon_\delta$, we find

$$0 \le \|V_\delta - \chi\|_2^2 = (1-\delta)^2 + 1 - 2(1-\varepsilon_\delta) \le C\varepsilon_\delta$$

and hence $(10)_1$, which in turn implies $(10)_2$ thanks to $R_\delta - A\chi = A(V_\delta - \chi)$ and $(7)_2$. By (6) we also have

$$\mu_\delta^2\, V_\delta(0) = \varepsilon_\delta^{m+2}\int_{-\frac12}^{+\frac12}\Phi'(R_\delta(x)) \le \varepsilon_\delta^{m+2}\Phi'(1-\varepsilon_\delta) \le C\varepsilon_\delta\,,$$

and this provides the upper bound in (11) since the unimodality of V_δ combined with $(10)_1$ guarantees that $\liminf_{\delta\to0} V_\delta(0) > 0$. To obtain the corresponding lower bound, we notice that (1) ensures

$$\|R_\delta''\|_\infty \le \frac{4\|\Phi'(R_\delta)\|_\infty}{\sigma_\delta} \le \frac{C}{\sigma_\delta\varepsilon_\delta^{m+1}}$$

and hence

$$R_\delta(x) \ge 1 - C\varepsilon_\delta \quad \text{for all} \quad |x| < \sqrt{\sigma_\delta\varepsilon_\delta^{m+2}} = \mu_\delta$$

due to $R_\delta'(0) = 0$ and $R_\delta(0) = 1 - \varepsilon_\delta$. Combining this with (13) we obtain

$$\frac{\mu_\delta^2}{\varepsilon_\delta^{m+2}}(1-\delta)^2 \ge \int_{-\mu_\delta}^{+\mu_\delta}\Phi'(R_\delta(x))R_\delta(x)\,\mathrm{d}x \ge c\frac{\mu_\delta}{\varepsilon_\delta^{m+1}},$$

and the proof of (11) is complete. Finally, the properties of Φ imply

$$p_\delta = \mathcal{P}(V_\delta) \le \varepsilon_\delta^{-m}\|R_\delta\|_2^2 \le C\varepsilon_\delta^{-m}$$

so $\varepsilon_\delta \to 0$ is a consequence of $p_\delta \to \infty$. □

1.4. Justification of Assumption 1

We briefly sketch how Assumption 1 can be justified using a constrained optimization approach. All key arguments are presented in [6] for non-singular potentials Φ but can easily be adapted to the potential (3). At the end of this section we also discuss the results from [12] and [3].

The variational approach from [6] is based on the potential energy functional (8), which is convex and Gâteaux differentiable on the open unit ball in $\mathsf{L}^2(\mathbb{R})$; the derivative of $\mathcal{P}$ is given by

$$\partial_V\mathcal{P}(V) = A\Phi'(AV)\,,$$

so the traveling wave equation (6) is equivalent to $\sigma V = \partial_V \mathcal{P}(V)$. We further introduce the cone $\mathcal{C}$ of all L^2-functions that are even, unimodal and nonnegative, i.e. we set

$$\mathcal{C} := \{V \in \mathsf{L}^2(\mathbb{R}) : 0 \le V(x) \le V(y) = V(-y) \text{ for almost all } x \le y \le 0\}\,.$$

The key observation is that solitary waves as in Assumption 1 can be constructed as solutions to the constrained optimization problem

$$\begin{aligned}&\text{Maximize } \mathcal{P} \text{ under the norm constraint } \|V\|_2 = 1-\delta \text{ and}\\ &\text{the shape constraint } V \in \mathcal{C}.\end{aligned} \tag{14}$$

In the existence proof one has to ensure that maximizers do in fact exist and that the shape constraint does not contribute to the Euler-Lagrange equation for the maximizer. With respect to the latter issue we introduce the *improvement operator*

$$\mathcal{T}_\delta(V) := (1-\delta)\frac{A\Phi'(AV)}{\|A\Phi'(AV)\|_2}\,.$$

We are now able to describe the key arguments in the variational existence proof for solitary waves with profiles in $\mathcal{C}$.

Lemma 4 (three ingredients)**.**

(i) Since Φ is strictly super-quadratic, each maximizing sequence for (14) is strongly compact.

(ii) The cone $\mathcal{C}$ is invariant under the actions of both the convolution operator A and the superposition operator Φ'. In particular, $V \in \mathcal{C}$ implies $R \in \mathcal{C}$ and $\mathcal{T}_\delta(V) \in \mathcal{C}$.

(iii) We have $\mathcal{P}(\mathcal{T}_\delta(V)) \ge \mathcal{P}(V)$, where the equality sign holds if an only if V is a fixed point of $\mathcal{T}_\delta$.

Sketch of the proof. The main steps can be summarized as follows:

(i) The assertion follows by a variant of the *Concentration Compactness Principle*.

(ii) The invariance properties can be checked by straight forward calculations.

(iii) Since Φ is convex, we have

$$\begin{aligned}\mathcal{P}(\mathcal{T}_\delta(V)) - \mathcal{P}(V) &\ge \langle \partial_V \mathcal{P}(V),\, \mathcal{T}_\delta(V) - V\rangle\\ &= \sigma(V)\langle \mathcal{T}_\delta(V),\, \mathcal{T}_\delta(V) - V\rangle\\ &= \tfrac{1}{2}\sigma(V)\|\mathcal{T}_\delta(V) - V\|_2^2\,,\end{aligned}$$

where we used $\|\mathcal{T}_\delta(V)\|_2 = \|V\|_2 = 1-\delta$ and that $\sigma(V) := \|A\Phi'(AV)\|_2/(1-\delta)$ is well defined as long as $\mathcal{P}(V) > 0$.

The details can be found in [6]. □

Corollary 5 (variational existence proof)**.** *For any $0 < \delta < 1$, there exists a solitary wave with $V_\delta, R_\delta \in \mathcal{C} \cap \mathsf{BC}^\infty(\mathbb{R})$ and $\|V_\delta\| = 1-\delta$. Moreover, we have $\mathcal{P}(V_\delta) \to \infty$ as $\delta \to 0$.*

Proof. The existence of a solution V_δ to (14) can be established by the *Direct Method* due to the compactness result from Lemma 4 and since $\mathcal{P}$ is strongly continuous. Moreover, any maximizer V_δ satisfies

$$\mathcal{P}(V_\delta) \geq \mathcal{P}(\mathcal{T}_\delta(V_\delta))\,, \qquad \mathcal{P}(V_\delta) \geq \mathcal{P}((1-\delta)\,\chi)\,.$$

The first estimate implies $\mathcal{P}(V_\delta) = \mathcal{P}(T_\delta(V_\delta))$, so V_δ satisfies the traveling wave equation (6) and is therefore smooth. We also compute

$$\mathcal{P}((1-\delta)\chi) = 2\int_0^1 \Phi((1-\delta)(1-x))\,\mathrm{d}x \quad \xrightarrow{\delta\to 0} \quad +\infty$$

and the proof is complete. □

The improvement operator $\mathcal{T}_\delta$ can also be used to compute solitary waves numerically. In fact, imposing homogeneous Dirichlet boundary conditions on a large but bounded and fine grid, the integral operator A can easily be discretized by Riemann sums. The resulting recursive scheme exhibits very good convergence properties; it has been applied to a wide range of potentials, see for instance [1, 6], and also been used to compute the numerical data displayed in Figures 1 and 2.

A different variational framework has been introduced in [5] and later been applied to the high-energy limit in [3]. The key idea there is to minimize the kinetic energy term $\frac{1}{2}\|V\|_2^2$ subject to a prescribed value of $p = \mathcal{P}(V)$. The results from [3] imply for the potential (3) that solitary waves converge as $p \to \infty$ to the limit function χ and satisfy Assumption 1 (though, strictly speaking, neither the unimodality nor the evenness of the profile functions R and V have been shown). In this context we emphasize again that uniqueness of Hamiltonian lattice waves is a notoriously difficult and an almost completely open problem. It is commonly believed that all variational and non-variational approaches provide – up to reparametrizations and for, say, convex potentials – the same family of solitary wave but there seems to be no proof so far.

A non-variational existence proof for solitary waves with high energy has been given in [12] using a carefully designed fixed-point argument for the (negative) distance profile R in the space of exponentially decaying functions. In our notations, the smallness parameter is ε and the waves are shown to satisfy $\|R - A\chi\| = O(\varepsilon)$ in some suitably chosen norm. We therefore expect that the waves constructed in [12] also satisfy Assumption 1, although the justification of the unimodality might be an issue.

2. Main result on the tip scaling of R_δ

Our first goal is to describe the asymptotic behavior of the distance profile R_δ near $x = 0$ by showing that it converges as $\delta \to 0$ under an appropriately defined rescaling to some nontrivial limit function. In view of theorem 2 and the numerical simulations from figure 2 we expect that both the variable x as well as the shifted amplitude variable $1 - R_\delta(\delta)$ must be scaled with certain powers of δ. A naive ansatz, however, does not

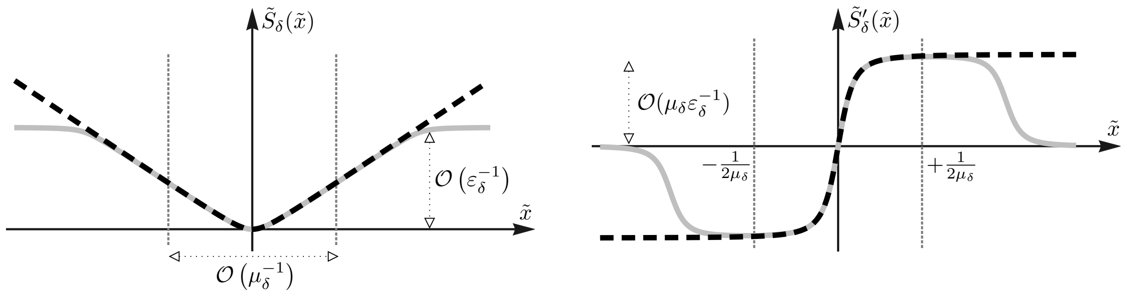

Figure 5. Cartoon of the tip scaling: The function $\tilde{S}_\delta$ and its derivative for $\delta > 0$ (gray, solid) and $\delta = 0$ (black, dashed). The dotted vertical lines enclose the symmetric interval J_δ which has length $1/\mu_\delta \gg 1$. The convergence $\tilde{S}_\delta \to \tilde{S}_0$ as $\delta \to 0$ implies $\mu_\delta \sim \varepsilon_\delta$, see Theorem 7 and Corollary 8.

work here because we lack a priori scaling relations between the small parameters δ, ε_δ, and μ_δ. For instance, if we would start with the rescaling

$$R_\delta(x) = 1 - \delta^{\gamma_1} \bar{R}_\delta(\delta^{\gamma_2} x)\,,$$

we could not eliminate σ_δ in the leading order equation. To overcome this problem we base our analysis on an implicit scaling, which magnifies the amplitude with ε_δ but defines the rescaled space variable by

$$\tilde{x} = \mu_\delta\, x\,.$$

In this way we obtain an explicit leading order equation that does not involve any unknown parameter and can hence be solved. Moreover, the corresponding solution finally allows us to identify the scaling relations between the different parameters; at the end it turns out that δ, ε_δ and μ_δ are all proportional to each other, see Corollary 13.

2.1. Implicit rescaling of R_δ

In order to derive asymptotic formulas for R_δ near $x = 0$, we define the rescaled distance profile

$$\tilde{S}_\delta(\tilde{x}) := \frac{R_\delta(0) - R_\delta(\mu_\delta \tilde{x})}{\varepsilon_\delta} = \frac{1 - \varepsilon_\delta - R_\delta(\mu_\delta \tilde{x})}{\varepsilon_\delta} \tag{15}$$

and obtain an even function which satisfies

$$\tilde{S}_\delta(0) = \tilde{S}'_\delta(0) = 0\,, \qquad \tilde{S}_\delta(\tilde{x}) = \tilde{S}_\delta(-\tilde{x})\,, \tag{16}$$

see Figure 5 for an illustration. We also introduce the auxiliary functions

$$\tilde{F}_\delta(\tilde{x}) := \varepsilon_\delta^{m+1} \Phi'\Big(R_\delta(\mu_\delta \tilde{x})\Big) \tag{17}$$

$$\tilde{G}_\delta(\tilde{x}) := \varepsilon_\delta^{m+1} \Phi'\Big(R_\delta(-1 + \mu_\delta \tilde{x})\Big), \tag{18}$$

as well as the intervals

$$I_\delta := \left[0, \frac{1}{2\mu_\delta}\right]\,, \qquad J_\delta = (-I_\delta) \cup I_\delta$$

and study the limit of $\tilde{S}_\delta$ restricted to J_δ.

Employing the identity $(1)_1$ as well as (4) we readily verify

$$\begin{aligned}\tilde{S}''_\delta(\tilde{x}) &= \frac{\mu_\delta^2}{\varepsilon_\delta}\Big(V'_\delta\big(\mu_\delta\tilde{x}-\tfrac12\big)-V'_\delta\big(\mu_\delta\tilde{x}+\tfrac12\big)\Big)\\ &= \sigma_\delta\varepsilon_\delta^{m+1}\Big(V'_\delta\big(\mu_\delta\tilde{x}-\tfrac12\big)-V'_\delta\big(\mu_\delta\tilde{x}+\tfrac12\big)\Big)\end{aligned}$$

and by $(1)_2$ we arrive at

$$\tilde{S}''_\delta(\tilde{x}) = 2\tilde{F}_\delta(\tilde{x}) - \tilde{G}_\delta(\tilde{x}) - \tilde{G}_\delta(-\tilde{x})\,, \tag{19}$$

where we used that $R_\delta(1+\mu_\delta\tilde{x}) = R_\delta(-1-\mu_\delta\tilde{x})$. The definition of $\tilde{G}_\delta$ combined with the unimodality of R_δ implies

$$0 \le \tilde{G}_\delta(\tilde{x}) \le \tilde{G}_\delta\left(\frac{1}{2\mu_\delta}\right) = \varepsilon_\delta^{m+1}\Phi'\big(R_\delta\big(\tfrac12\big)\big) \le C\varepsilon_\delta^{m+1} \quad \text{for all } \tilde{x}\in J_\delta \tag{20}$$

and from (17) we get

$$\tilde{F}_\delta(\tilde{x}) = \frac{\Psi\Big(\varepsilon_\delta + \varepsilon_\delta\tilde{S}_\delta(\tilde{x})\Big)}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+1}}\,, \tag{21}$$

where the function $\Psi : [0,\,1] \to \mathbb{R}_+$ with

$$\Psi(s) := \Phi'(1-s)s^{m+1} \quad \text{for} \quad 0 < s \le 1\,, \qquad \Psi(0) := \lim_{s\searrow 0}\Psi(s) = \frac{1}{m+1}$$

is smooth and positive. In particular, on the interval J_δ we find

$$\tilde{S}''_\delta(\tilde{x}) \approx 2\tilde{F}_\delta(\tilde{x}) \approx \frac{2}{m+1}\cdot\frac{1}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+1}}\,,$$

and conclude that $\tilde{S}_\delta$ satisfies the initial value problem (5) from the introduction up to small error terms.

Lemma 6 (solution of the limit problem)**.** *The initial value problem (5) has a unique solution which is even, nonnegative, and convex. This solution $\tilde{S}_0$ grows linearly for $\tilde{x}\to\pm\infty$ as it satisfies*

$$\Big|\tilde{S}'_0(\tilde{x}) - \overline{\mu}\,\mathrm{sgn}(\tilde{x})\Big| \le \frac{C}{(1+\tilde{x})^m} \tag{22}$$

and

$$\Big|\tilde{x}\tilde{S}'_0(\tilde{x}) - \tilde{S}_0(\tilde{x}) - \overline{\kappa}\Big| \le \frac{C}{(1+\tilde{x})^{m-1}} \tag{23}$$

for all $\tilde{x}\in\mathbb{R}$ with

$$\overline{\mu} := \frac{2}{\sqrt{m(m+1)}}\,, \quad \overline{\kappa} := \int_0^\infty \tilde{x}\,\tilde{S}''_0(\tilde{x})\,\mathrm{d}\tilde{x}\,, \quad \overline{\eta} := \int_0^\infty \tilde{S}_0(\tilde{x})\,\tilde{S}''_0(\tilde{x})\,\mathrm{d}\tilde{x} \tag{24}$$

and some constant C which depends only on m.

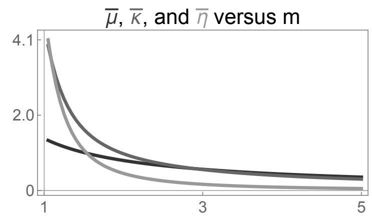

Figure 6. Numerical values for the constants $\overline{\mu}$, $\overline{\kappa}$, and $\overline{\eta}$ from (24), which provide the leading and the next-to-leading order terms in the scaling relations between ε_δ, μ_δ, and σ_δ, see Corollary (13).

Proof. The planar and autonomous Hamiltonian ODE $(5)_1$ admits the conserved quantity

$$E_{\rm tot}(\tilde{x}) := \tfrac{1}{2}\Big(\tilde{S}_0'(\tilde{x})\Big)^2 + E_{\rm pot}(\tilde{x})\,, \qquad E_{\rm pot}(\tilde{x}) := \frac{\tfrac{1}{2}\overline{\mu}^2}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^m}$$

with value $E_{\rm tot}(\tilde{x}) = E_{\rm tot}(0) = \tfrac{1}{2}\overline{\mu}^2$ for all $\tilde{x}$. A simple phase plane analysis reveals that $\tilde{S}$ is even and that both $\tilde{S}_0$ and $\tilde{S}_0'$ are strictly increasing for $\tilde{x} > 0$, see Figure 4 for an illustration. In particular, we have

$$\tilde{S}_0'(\tilde{x}) \quad \xrightarrow{\tilde{x}\to\infty} \quad \sqrt{2E_{\rm tot}(0)} = \overline{\mu} > 0$$

so the conservation law implies

$$\begin{aligned}\Big|\tilde{S}_0'(\tilde{x}) - \sqrt{2E_{\rm tot}(0)}\Big| \; &= \; \Big|\sqrt{2E_{\rm tot}(0) - 2E_{\rm pot}(\tilde{x})} - \sqrt{2E_{\rm tot}(0)}\Big| \\ &\leq C\,E_{\rm pot}(\tilde{x}) \leq \frac{C}{(1+\tilde{x})^m}\,,\end{aligned}$$

and hence (22). Moreover, the even function $\tilde{K}_0$ with $\tilde{K}_0(\tilde{x}) := \tilde{x}\tilde{S}_0'(\tilde{x}) - \tilde{S}_0(\tilde{x})$ satisfies

$$0 \leq \tilde{K}_0'(\tilde{x}) = \tilde{x}\tilde{S}_0''(\tilde{x}) \leq \frac{C}{(1+\tilde{x})^m} \qquad \text{for all} \quad \tilde{x} > 0\,,$$

so $\tilde{K}_0'$ is integrable due to $m > 1$. The constant $\overline{\kappa} = \lim_{\tilde{x}\to\infty}\tilde{K}_0(\tilde{x})$ is therefore well-defined and (23) follows immediately from the estimate for $\tilde{K}_0'(\tilde{x})$. Finally, $\overline{\eta}$ is well-defined since the integrand is continuous and decays as $\tilde{x}^{-m}$ for $\tilde{x}\to\infty$. □

There seems to be no simple way to compute the constants $\overline{\kappa}$ and $\overline{\eta}$ as functions of m but numerical values are presented in Figure 6.

2.2. Asymptotic formulas for $\tilde{S}_\delta$

We now able to formulate and prove our main asymptotic result.

Theorem 7 (asymptotics of $\tilde{S}_\delta$)**.** *Any function $\tilde{S}_\delta$ is strictly increasing and convex on I_δ. Moreover, the estimates*

$$\sup_{\tilde{x}\in J_\delta}\Big|\tilde{S}_\delta(\tilde{x}) - \tilde{S}_0(\tilde{x})\Big| \leq C\varepsilon_\delta^{m-1} \tag{25}$$

and

$$\sup_{\tilde{x}\in J_\delta}\Big|\tilde{S}_\delta'(\tilde{x})-\tilde{S}_0'(\tilde{x})\Big|\le C\varepsilon_\delta^m\,,\qquad \sup_{\tilde{x}\in J_\delta}\Big|\tilde{S}_\delta''(\tilde{x})-\tilde{S}_0''(\tilde{x})\Big|\le C\varepsilon_\delta^{m+1} \tag{26}$$

hold for all $0<\delta<1$ *and a constant* C *independent of* δ.

Proof. Since $\tilde{S}_\delta$ and $\tilde{S}_0$ are even functions it suffices to consider $\tilde{x}\in I_\delta$.

Dynamics of $\tilde{S}_\delta$: By (21) and due to $\Psi(0)-\Psi(s)=Cs^{m+1}$ we have

$$\left|\tilde{F}_\delta(\tilde{x})-\frac{\Psi(0)}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+1}}\right|\le C\varepsilon_\delta^{m+1}\,,$$

and in view of (21) and (20) we conclude that $\tilde{S}_\delta$ satisfies on the interval I_δ the ODE

$$\tilde{S}_\delta''(\tilde{x})=\frac{\Psi(0)}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+1}}+\varepsilon_\delta^{m+1}h_\delta(\tilde{x})\qquad\text{with}\quad |h_\delta(\tilde{x})|\le C\,. \tag{27}$$

Standard ODE arguments now imply

$$\sup_{0\le\tilde{x}\le\tilde{x}_*}\Big(\Big|\tilde{S}_\delta(\tilde{x})-\tilde{S}_0(\tilde{x})\Big|+\Big|\tilde{S}_\delta'(\tilde{x})-\tilde{S}_0'(\tilde{x})\Big|\Big)\le C\varepsilon_\delta^{m+1} \tag{28}$$

for any fixed $\tilde{x}_*>0$, where C depends on $\tilde{x}_*$.

Properties of $\tilde{S}_\delta$: The monotonicity of both $\tilde{S}_\delta$ and $\tilde{S}_\delta'$ on I_δ follows directly from the unimodality of R_δ, V_δ and the definition (15). In particular, $\tilde{S}_\delta$ is convex on I_δ. We therefore have

$$\tilde{S}_\delta(\tilde{x})\ge\tilde{S}_\delta(\tilde{x}_*)+\tilde{S}_\delta(\tilde{x}_*)'\,(\tilde{x}-\tilde{x}_*)\,,$$

and choosing $\tilde{x}_*>0$ sufficiently close to 0 we find a constant c such that

$$1+\tilde{S}_\delta(x)\ge c(1+\tilde{x}) \tag{29}$$

holds for all $\tilde{x}\in I_\delta$, where we used that $\lim_{\delta\to 0}\tilde{S}_\delta'(\tilde{x}_*)=\tilde{S}_0'(\tilde{x}_*)>0$ is implied by (28). Notice that (29) holds also for $\delta=0$.

Estimates for $\tilde{S}_\delta$: By (5) and (27) – and using both the Mean Value Theorem as well as (29) – we obtain

$$\Big|\tilde{S}_\delta''(\tilde{x})-\tilde{S}_0''(\tilde{x})\Big|\le\frac{C}{(1+\tilde{x})^{m+2}}\Big|\tilde{S}_\delta(\tilde{x})-\tilde{S}_0(\tilde{x})\Big|+C\varepsilon_\delta^{m+1}\,. \tag{30}$$

Integration with respect to $\tilde{x}$ yields

$$\Big|\tilde{S}_\delta'(\tilde{x})-\tilde{S}_0'(\tilde{x})\Big|\le C\varepsilon_\delta^{m+1}\tilde{x}+\int_0^{\tilde{x}}\frac{C}{(1+\tilde{y})^{m+2}}\int_0^{\tilde{y}}\Big|\tilde{S}_\delta'(\tilde{z})-\tilde{S}_0'(\tilde{z})\Big|\,\mathrm{d}\tilde{z}\,\mathrm{d}\tilde{y}$$

since (16) ensures that

$$\tilde{S}_\delta(0)=\tilde{S}_0(0)=0\,,\qquad \tilde{S}_\delta'(0)=\tilde{S}_0'(0)=0\,,$$

and a direct computation reveals

$$\Big|\tilde{S}_\delta'(\tilde{x})-\tilde{S}_0'(\tilde{x})\Big|\le C\varepsilon_\delta^{m+1}\tilde{x}+\int_0^{\tilde{x}}\frac{C}{(1+\tilde{z})^{m+1}}\Big|\tilde{S}_\delta'(\tilde{z})-\tilde{S}_0'(\tilde{z})\Big|\,\mathrm{d}\tilde{z}\,.$$

Employing the Gronwall Lemma for $\tilde{x} \geq 0$ we obtain

$$\left|\tilde{S}_\delta'(\tilde{x}) - \tilde{S}_0'(\tilde{x})\right| \leq C\varepsilon_\delta^{m+1}\tilde{x}\exp\left(\int_0^{\tilde{x}} \frac{C}{(1+\tilde{z})^{m+1}}\,\mathrm{d}\tilde{z}\right) \leq C\varepsilon_\delta^{m+1}\tilde{x}\,, \tag{31}$$

and using $\tilde{x} \leq 1/(2\tilde{\mu}_\delta)$ as well as the lower bound for μ_δ from Lemma 3 we arrive at $(26)_1$. Moreover, integrating (31) with respect to $\tilde{x}$ gives

$$\left|\tilde{S}_\delta(\tilde{x}) - \tilde{S}_0(\tilde{x})\right| \leq C\varepsilon_\delta^{m+1}\tilde{x}^2$$

and hence (25). In combination with (30) we further get

$$\left|\tilde{S}_\delta''(\tilde{x}) - \tilde{S}_0''(\tilde{x})\right| \leq C\varepsilon_\delta^{m+1}\left(\frac{\tilde{x}^2}{(1+\tilde{x})^{m+2}} + 1\right),$$

which in turn provides $(26)_2$. □

A first consequence of Theorem 7 are leading order expressions for μ_δ and hence for σ_δ; below we improve this result by specifying the next-to-leading order corrections in Corollary 13.

Corollary 8 (convergence of μ_δ and σ_δ)**.**

$$\frac{\mu_\delta}{\varepsilon_\delta} \xrightarrow{\delta\to 0} \overline{\mu}, \qquad \sigma_\delta\varepsilon_\delta^m \xrightarrow{\delta\to 0} \overline{\mu}^2\,. \tag{32}$$

Proof. By construction – see (15) – and Lemma 3 we have

$$\varepsilon_\delta\,\tilde{S}_\delta\left(\frac{1}{2\mu_\delta}\right) = 1 - \varepsilon_\delta - R_\delta\left(\tfrac{1}{2}\right) \xrightarrow{\delta\to 0} \tfrac{1}{2}\,.$$

and in view of Theorem 7 we conclude that

$$2\,\varepsilon_\delta\,\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right) \xrightarrow{\delta\to 0} 1\,.$$

On the other hand, the estimates (22) and (23) evaluated at $\tilde{x} = 1/(2\mu_\delta)$ imply

$$2\,\mu_\delta\,\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right) \xrightarrow{\delta\to 0} \overline{\mu}\,,$$

so the claim for μ_δ follows immediately. The convergence result for σ_δ is a consequence of (4). □

3. Further asymptotic formulas

In this section we exploit the asymptotic results on the tip scaling and derive approximation formulas for the foot and transition scaling, the tails of the profile functions, and for the scaling relations between δ, ε_δ, and σ_δ. In this way we obtain a complete set of asymptotic formulas which finally allows us to extract all relevant information on the limit $\delta \to 0$ from the function $\tilde{S}_0$ only. We also sketch a possible application of these formulas, namely the study of the linearized eigenvalue problem.

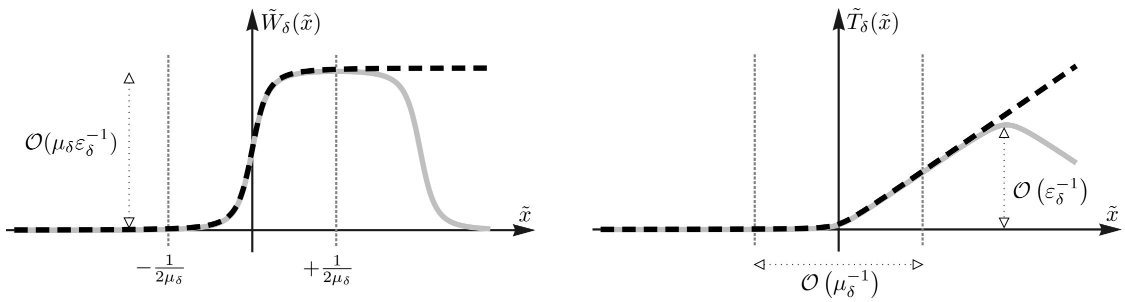

Figure 7. Cartoon of the tip and the foot scaling: The functions $\tilde{W}_\delta$ and $\tilde{T}_\delta$ for $\delta > 0$ (gray, solid) and $\delta = 0$ (black, dashed). The limits $\tilde{W}_0$ and $\tilde{T}_0$ can be computed from $\tilde{S}_0$, see (34) and (45).

3.1. Transition scaling of V_δ

To describe the jump-like behavior of V_δ near $x = \pm\frac{1}{2}$ we introduce the rescaled profiles $\tilde{W}_\delta$ by

$$\tilde{W}_\delta(\tilde{x}) := \frac{\mu_\delta}{\varepsilon_\delta} V_\delta\big(-\tfrac{1}{2} + \mu_\delta \tilde{x}\big) \tag{33}$$

and refer to Figure 7 for an illustration. The key observation for the asymptotics of $\tilde{W}_\delta$ is the approximation

$$\tilde{W}'_\delta(\tilde{x}) \approx \tilde{F}_\delta(\tilde{x}) \approx \tfrac{1}{2}\tilde{S}''_\delta(\tilde{x})\,,$$

and hence we are able to prove the convergence of $\tilde{W}_\delta$ to

$$\tilde{W}_0(\tilde{x}) := \tfrac{1}{2}\Big(\tilde{S}'_0(\tilde{x}) + \overline{\mu}\Big)\,, \tag{34}$$

where $\tilde{S}_0$ is the ODE solution from the tip scaling and defined in (5).

Theorem 9 (convergence under the transition scaling)**.** *We have*

$$\sup_{x\in J_\delta} |\tilde{W}_\delta(\tilde{x}) - \tilde{W}_0(\tilde{x})| \le C\varepsilon_\delta^m\,, \qquad \sup_{x\in J_\delta} |\tilde{W}'_\delta(\tilde{x}) - \tilde{W}'_0(\tilde{x})| \le C\varepsilon_\delta^{m+1} \tag{35}$$

for some constant C independent of δ.

Proof. *Uniform estimates for the derivative*: By (33), the traveling wave equation (1), and the definition of μ_δ in (4) we have

$$\begin{aligned}\tilde{W}'_\delta(\tilde{x}) &= \frac{\mu_\delta^2}{\varepsilon_\delta} V'_\delta\big(-\tfrac{1}{2} + \mu_\delta\tilde{x}\big)\\ &= \sigma_\delta \varepsilon_\delta^{m+1} V'_\delta\big(-\tfrac{1}{2} + \mu_\delta\tilde{x}\big)\\ &= \varepsilon_\delta^{m+1}\Big(\Phi'(R_\delta(\mu_\delta\tilde{x})) - \Phi'(R_\delta(-1+\mu_\delta\tilde{x}))\Big)\,,\end{aligned}$$

and from (17), (18) as well as (19) we infer that

$$\tilde{W}'_\delta(\tilde{x}) = \tilde{F}_\delta(\tilde{x}) - \tilde{G}_\delta(\tilde{x}) = \tfrac{1}{2}\Big(\tilde{S}''_\delta(\tilde{x}) - \tilde{G}_\delta(\tilde{x}) + \tilde{G}_\delta(-\tilde{x})\Big)\,.$$

The estimate $(35)_2$ is now a direct consequence of (20) and Theorem 7.

Pointwise estimate at $\tilde{x} = 1/(2\mu_\delta)$: Using (33), the traveling wave equation (6), and the unimodality of R_δ we obtain

$$\frac{\varepsilon_\delta}{\mu_\delta}\tilde{W}_\delta\left(\frac{1}{2\mu_\delta}\right) = V_\delta(0) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{\Phi'(R_\delta(x))}{\sigma_\delta}\,\mathrm{d}x$$
$$= \frac{2}{\sigma_\delta\,\varepsilon_\delta^{m+1}}\int_0^{+\frac{1}{2}} \varepsilon_\delta^{m+1}\,\Phi'(R_\delta(x))\,\mathrm{d}x\,.$$

Thanks to (4), (17), and (19) we thus conclude

$$\tilde{W}_\delta\left(\frac{1}{2\mu_\delta}\right) = \frac{\mu_\delta^2}{\sigma_\delta\,\varepsilon_\delta^{m+2}}\int_{I_\delta} 2\,\tilde{F}_\delta(\tilde{x})\,\mathrm{d}\tilde{x} = \int_{I_\delta} 2\,\tilde{F}_\delta(\tilde{x})\,\mathrm{d}\tilde{x}$$
$$= \int_{I_\delta} \tilde{S}''_\delta(\tilde{x}) - \tilde{G}_\delta(\tilde{x}) - \tilde{G}_\delta(-\tilde{x})\,\mathrm{d}\tilde{x}$$

and since $\int_{I_\delta}\tilde{G}_\delta(\pm\tilde{x})\,\mathrm{d}\tilde{x} = O(\varepsilon_\delta^m)$ holds according to (20) and Corollary 8, we obtain

$$\tilde{W}_\delta\left(\frac{1}{2\mu_\delta}\right) = S'_\delta\left(\frac{1}{2\mu_\delta}\right) + O(\varepsilon_\delta^m)\,,$$

Lemma 6 combined with Theorem 7 and (32) finally provides

$$\tilde{W}_\delta\left(\frac{1}{2\mu_\delta}\right) = \overline{\mu} + O(\varepsilon_\delta^m) = \tilde{W}_0\left(\frac{1}{2\mu_\delta}\right) + O(\varepsilon_\delta^m)\,,$$

so $(35)_1$ follows from $(35)_2$. □

3.2. Tail estimates

We complement our previous results by estimates for the tails of the profiles V_δ and R_δ. The derivation of those exploits the exponential decay with respect to x, which we establish by adapting an idea from [7].

Theorem 10 (tail estimates for V_δ and R_δ)**.** *The estimates*

$$\sup_{|x|\geq 1} V_\delta(x) \;+\; \int_{|x|\geq 1} V_\delta(x)\,\mathrm{d}x \;\leq\; C\varepsilon_\delta^m$$

and

$$\sup_{|x|\geq \frac{3}{2}} R_\delta(x) \;+\; \int_{|x|\geq \frac{3}{2}} R_\delta(x)\,\mathrm{d}x \;\leq\; C\varepsilon_\delta^m$$

hold for some constant C independent of δ.

Proof. Since V_δ is unimodal and nonnegative by Assumption 1, we have

$$0 \leq V_\delta(x) \leq V_\delta(-1) \quad \text{for all} \quad x \leq -1. \tag{36}$$

Moreover, the properties of the convolution operator A combined with (6) imply

$$R_\delta(x) \leq V_\delta\big(x + \tfrac{1}{2}\big)\,, \quad V_\delta(x) \leq \frac{\Phi'\Big(R_\delta\big(x + \tfrac{1}{2}\big)\Big)}{\sigma_\delta} \quad \text{for all} \quad x < -1 \tag{37}$$

and we infer that

$$V_\delta(x) \le \frac{\Phi'(V_\delta(x+1))}{\sigma_\delta} \le C{\varepsilon_\delta}^m V_\delta(x+1) \quad \text{for all} \quad x < -2 \tag{38}$$

where we used Corollary 8, the properties of Φ', and that the unimodality of V_δ guarantees

$$2V_\delta(-1)^2 \le \|V_\delta\|_2^2 = (1-\delta)^2 \qquad \text{and hence} \qquad V_\delta(-1) \le 1/\sqrt{2} < 1\,.$$

By iteration of (38) we obtain

$$V_\delta(x-n) \le (C\varepsilon_\delta^m)^n V_\delta(x) \qquad \text{for all} \quad x < -1\,,\ n \in \mathbb{N}$$

and conclude that V_δ decays exponentially with rate

$$\lambda_\delta \ge m\,|\ln\varepsilon_\delta|\,(1+o(1))$$

and satisfies

$$\int\limits_{-\infty}^{-1} V_\delta(x)\,\mathrm{d}x \le CV_\delta(-1)\,. \tag{39}$$

Finally, (33) and Theorem 9 ensure that

$$\begin{aligned} V_\delta(-1) &= \frac{\varepsilon_\delta}{\mu_\delta}\tilde{W}_\delta\left(-\frac{1}{2\mu_\delta}\right) \\ &= \frac{\varepsilon_\delta}{\mu_\delta}\left(\frac{\overline{\mu}+\tilde{S}_0'\left(-\frac{1}{2\mu_\delta}\right)}{2} + O(\varepsilon_\delta^m)\right) = O(\varepsilon_\delta^m) \end{aligned} \tag{40}$$

where the last identity stems from (22) and (32). The desired estimates for V_δ are now direct consequences of (36), (39), and (40), and imply the claim for R_δ thanks to $(37)_1$. □

Notice that the exponential decay rate in the proof of Theorem 10 is asymptotically optimal for small δ. In fact, after linearization of Φ' in 0 we find the tail identity

$$\sigma_\delta V_\delta \approx A^2 V_\delta\,,$$

and the usual exponential ansatz predicts that the exact decay rate λ_δ is the positive solution to the transcendental equation

$$\frac{\sinh(\lambda_\delta/2)}{\lambda_\delta/2} = \sqrt{\sigma_\delta} = \overline{\mu}\varepsilon_\delta^{-m/2}\Big(1+O(\varepsilon_\delta)\Big)\,.$$

In particular, λ_δ is large for small δ and satisfies $\lambda_\delta = m\,|\ln\varepsilon_\delta| + O(\ln|\ln\varepsilon_\delta|)$.

3.3. Relations between the parameters δ, ε_δ, and μ_δ

In this section we identify the scaling relations between the small quantities

$$\delta, \quad \varepsilon_\delta\,, \quad \mu_\delta$$

and start with two auxiliary results.

Lemma 11 (scaling relation between μ_δ and ε_δ)**.** *The formula*

$$\mu_\delta = \frac{\overline{\mu}\,\varepsilon_\delta}{1+\varepsilon_\delta(\overline{\kappa}-1)} + O\big(\varepsilon_\delta^{m+1}\big)$$

holds for all $0<\delta<1$.

Proof. Our starting point is the identity

$$\begin{aligned} 1-\varepsilon_\delta = R_\delta(0) &= 2\int_{-1/2}^{0} V_\delta(x)\,\mathrm{d}x \\ &= 2\,\mu_\delta\int_0^{1/(2\mu_\delta)} V_\delta\left(-\frac{1}{2}+\tilde{\mu}\tilde{x}\right)\mathrm{d}\tilde{x} = 2\,\varepsilon_\delta\int_0^{1/(2\mu_\delta)} \tilde{W}_\delta(\tilde{x})\,\mathrm{d}\tilde{x} \end{aligned}$$

which follows from (6), Assumption 1, and (33). Theorem 9 now yields

$$\begin{aligned} 1-\varepsilon_\delta &= 2\,\varepsilon_\delta\int_0^{1/(2\mu_\delta)} \tilde{W}_0(\tilde{x})\,\mathrm{d}x + O(\varepsilon_\delta^m) \\ &= \frac{\varepsilon_\delta}{\mu_\delta}\left(\frac{\overline{\mu}}{2}+\mu_\delta\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right)\right) + O(\varepsilon_\delta^m)\,, \end{aligned}$$

while (23) and (32) provide

$$\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right) = \frac{1}{2\mu_\delta}\tilde{S}_0'\left(\frac{1}{2\mu_\delta}\right) - \overline{\kappa} + O\big(\mu_\delta^{m-1}\big) = \frac{\overline{\mu}}{2\mu_\delta} - \overline{\kappa} + O\big(\varepsilon_\delta^{m-1}\big)\,.$$

The combination of the latter two formulas yields

$$1-\varepsilon_\delta = \frac{\varepsilon_\delta}{\mu_\delta}\Big(\overline{\mu}-\mu_\delta\overline{\kappa}\Big) + O(\varepsilon_\delta^m)\,,$$

and rearranging terms we find the desired result. □

Lemma 12 (scaling relation between ε_δ and δ)**.** *We have*

$$\delta = 1-\sqrt{\Big(1+\varepsilon_\delta(\overline{\kappa}-1)\Big)\left(1-\varepsilon_\delta\left(1+\frac{\overline{\eta}}{\overline{\mu}}\right)\right)} + O(\varepsilon_\delta^m)\,,$$

where the right hand side is real-valued and positive for all sufficiently small $\delta>0$.

Proof. From Assumption 1 and the tail estimates in Theorem 10 we derive

$$(1-\delta)^2 = \int_{\mathbb{R}} V_\delta(x)^2\,\mathrm{d}x = 2\int_{-1}^{0} V_\delta(x)^2\,\mathrm{d}x + O(\varepsilon_\delta^m)\,,$$

and (33) gives

$$\int_{-1}^{0} V_\delta(x)^2\,\mathrm{d}x = \mu_\delta\int_{J_\delta} V_\delta\left(-\frac{1}{2}+\mu_\delta\tilde{x}\right)^2\mathrm{d}\tilde{x} = \frac{\varepsilon_\delta^2}{\mu_\delta}\int_{J_\delta}\tilde{W}_\delta(\tilde{x})^2\,\mathrm{d}\tilde{x}\,.$$

Therefore – and thanks to Theorem 9 – we get

$$\begin{aligned}(1-\delta)^2 &= 2\,\frac{\varepsilon_\delta^2}{\mu_\delta}\int\limits_{J_\delta}\tilde{W}_0(\tilde{x})^2\,\mathrm{d}\tilde{x}+O(\varepsilon_\delta^m)\\ &= \frac{\overline{\mu}^2\varepsilon_\delta^2}{2\mu_\delta^2}+\frac{\varepsilon_\delta^2}{\mu_\delta}\int_{I_\delta}\tilde{S}_0'(\tilde{x})^2\,\mathrm{d}\tilde{x}+O(\varepsilon_\delta^m)\end{aligned} \tag{41}$$

where we used that $\tilde{S}_0'$ is an odd function, and integration by parts yields

$$\begin{aligned}\int_{I_\delta}\big(\tilde{S}_0'(\tilde{x})\big)^2\,\mathrm{d}\tilde{x} &= \tilde{S}_0\left(\frac{1}{2\mu_\delta}\right)\tilde{S}_0'\left(\frac{1}{2\mu_\delta}\right)-\int_{I_\delta}\tilde{S}_0(\tilde{x})\,\tilde{S}_0''(\tilde{x})\,\mathrm{d}\tilde{x}\\ &= \left(\frac{1}{2\mu_\delta}\tilde{S}_0'\left(\frac{1}{2\mu_\delta}\right)-\overline{\kappa}\right)\tilde{S}_0'\left(\frac{1}{2\mu_\delta}\right)-\overline{\eta}+O\big(\mu_\delta^{m-1}\big)\\ &= \left(\frac{\overline{\mu}}{2\mu_\delta}-\overline{\kappa}\right)\overline{\mu}-\overline{\eta}+O\big(\mu_\delta^{m-1}\big)\end{aligned}$$

thanks to the estimates and decay results from Lemma 6. In summary we find

$$(1-\delta)^2=\frac{\overline{\mu}^2\varepsilon_\delta^2}{\mu_\delta^2}-\frac{\varepsilon_\delta^2}{\mu_\delta}(\overline{\kappa}\,\overline{\mu}+\overline{\eta})+O(\varepsilon_\delta^m)$$

and eliminating μ_δ by Lemma 11 yields via

$$(1-\delta)^2=\Big(1+\varepsilon_\delta(\overline{\kappa}-1)\Big)^2-\varepsilon_\delta\Big(1+\varepsilon_\delta(\overline{\kappa}-1)\Big)\left(\overline{\kappa}+\frac{\overline{\eta}}{\overline{\mu}}\right)+O(\varepsilon_\delta^m) \tag{42}$$

the assertion. □

Lemma 11 and Lemma 12 provide explicit formulas for

$$\mu_\delta\sim\varepsilon_\delta \qquad \text{and} \qquad \delta\sim\varepsilon_\delta$$

in terms of ε_δ but there is a slight mismatch between both results since the error bounds in the formula for μ_δ are of higher order than those in the scaling law for δ. It is not clear, at least to the authors, whether this mismatch concerns the real error terms or just means that the bounds in Lemma 12 are less optimal than those in Lemma 11.

Our main result concerning the scaling relations between the different parameters can now be formulated as follows.

Corollary 13 (leading order scaling laws)**.** *The relation between μ_δ and ε_δ can be computed up to error terms of order $O\big(\varepsilon_\delta^{m+1}\big)$, while the scaling law between δ and ε_δ is determined up to order $O(\varepsilon_\delta^m)$ only. In particular, we have*

$$\begin{aligned}\mu_\delta &= \overline{\mu}\,\varepsilon_\delta+\overline{\mu}\,(1-\overline{\kappa})\,\varepsilon_\delta^2+o\big(\varepsilon_\delta^2\big)\,,\\ \sigma_\delta &= \overline{\mu}^2\varepsilon_\delta^{-m}+2\,\overline{\mu}^2\,(1-\overline{\kappa})\,\varepsilon_\delta^{-m+1}+o\big(\varepsilon_\delta^{-m+1}\big)\end{aligned}$$

for all $m>1$, as well as

$$\delta=\frac{2\,\overline{\mu}-\overline{\mu}\,\overline{\kappa}-\overline{\eta}}{2\,\overline{\mu}}\varepsilon_\delta+\frac{\overline{\mu}^2\,\overline{\kappa}^2+\overline{\eta}^2+2\,\overline{\mu}\,\overline{\kappa}\,\overline{\eta}}{8\,\overline{\mu}^2}\,\varepsilon_\delta^2+o\big(\varepsilon_\delta^2\big)$$

provided that $m>2$.

Proof. All assertions are provided by Lemma 11, Lemma 12, and formula (4). □

3.4. Foot scaling of R_δ

We study now the asymptotic behavior of R_δ near $x = \pm 1$. To this end we define

$$\tilde{T}_\delta(\tilde{x}) := \frac{R_\delta(-1+\mu_\delta \tilde{x})}{\varepsilon_\delta} \tag{43}$$

and find by direct calculations the identity

$$\tilde{T}''_\delta(\tilde{x}) = \tilde{F}_\delta(\tilde{x}) + \tilde{H}_\delta(\tilde{x}) - 2\tilde{G}_\delta(\tilde{x}) \approx \tfrac{1}{2}\tilde{S}''_\delta(\tilde{x}) \tag{44}$$

because both $\tilde{G}_\delta$ and

$$\tilde{H}_\delta(\tilde{x}) := \Phi'(R_\delta(-2+\mu_\delta\tilde{x}))$$

can be neglected on the interval I_δ. We further define

$$\tilde{T}_0(\tilde{x}) := \tfrac{1}{2}\Big(\tilde{S}_0(\tilde{x}) + \overline{\mu}\,\tilde{x} + \overline{\kappa}\Big) \tag{45}$$

and show that $\tilde{T}_\delta$ converges as $\delta \to 0$ to $\tilde{T}_0$, see Figure 7.

Lemma 14 (asymptotics of R_δ at $x = \pm\frac{1}{2}$ and for V_δ at $x = 0$)**.** *The terms* $R'_\delta\big(-\tfrac{1}{2}\big)$, $2R_\delta\big(\tfrac{1}{2}\big)$, *and* $V_\delta(0)$ *are identical to leading order in* δ. *More precisely, we have*

$$R'_\delta\big(\tfrac{1}{2}\big) = \frac{\varepsilon_\delta\,\overline{\mu}}{\mu_\delta} + O(\varepsilon_\delta^m) = (1+\varepsilon_\delta(\overline{\kappa}-1)) + O(\varepsilon_\delta^m)\,,$$

and

$$|2\,R_\delta\big(\tfrac{1}{2}\big) - R'_\delta\big(-\tfrac{1}{2}\big)| = O(\varepsilon_\delta^m)\,, \qquad |V_\delta(0) - R'_\delta\big(\text{-}\tfrac{1}{2}\big)| = O(\varepsilon_\delta^m)$$

for all $0 < \delta < 1$.

Proof. From (15), Lemma 6, Theorem 7, and (32) we infer

$$\begin{aligned} R'_\delta\big(-\tfrac{1}{2}\big) &= \,-\,R'_\delta\big(\tfrac{1}{2}\big) = \frac{\varepsilon_\delta}{\mu_\delta}\tilde{S}'_\delta\left(\frac{1}{2\mu_\delta}\right) \\ &= \frac{\varepsilon_\delta}{\mu_\delta}\tilde{S}'_0\left(\frac{1}{2\mu_\delta}\right) + O(\mu_\delta^m) = \frac{\varepsilon_\delta\,\overline{\mu}}{\mu_\delta} + O(\varepsilon_\delta^m) \end{aligned}$$

and similarly

$$\begin{aligned} R_\delta\big(\tfrac{1}{2}\big) &= 1 - \varepsilon_\delta - \varepsilon_\delta\tilde{S}_\delta\left(\frac{1}{2\mu_\delta}\right) = 1 - \varepsilon_\delta - \varepsilon_\delta\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right) + O(\mu_\delta^m) \\ &= 1 - \varepsilon_\delta - \varepsilon_\delta\left(\frac{1}{2\mu_\delta}\tilde{S}'_0\left(\frac{1}{2\mu_\delta}\right) - \overline{\kappa}\right) + O(\varepsilon_\delta^m) \\ &= 1 + \varepsilon_\delta(\overline{\kappa}-1) - \frac{\varepsilon_\delta\,\overline{\mu}}{2\,\mu_\delta} + O(\varepsilon_\delta^m)\,. \end{aligned}$$

Thanks to (33) and Theorem 9 we also find

$$V_\delta(0) = \frac{\varepsilon_\delta}{\mu_\delta}\tilde{W}_0\left(\frac{1}{2\mu_\delta}\right) + O(\varepsilon_\delta^m) = \frac{\varepsilon_\delta\,\overline{\mu}}{\mu_\delta} + O(\varepsilon_\delta^m)\,,$$

and the result follows from Lemma 11. □

Theorem 15 (convergence under the foot scaling)**.** *The estimate*

$$\sup_{\tilde{x}\in J_\delta}\Big(\varepsilon^2\,|\tilde{T}_\delta(\tilde{x})-\tilde{T}_0(\tilde{x})|+\varepsilon\,|\tilde{T}_\delta'(\tilde{x})-\tilde{T}_0'(\tilde{x})|+|\tilde{T}_\delta''(\tilde{x})-\tilde{T}_0''(\tilde{x})|\Big)\le C\varepsilon^{m+1}$$

holds with some constant C independent of δ. In particular, we have

$$R_\delta(\pm1)=\tfrac{1}{2}\overline{\kappa}\,\varepsilon_\delta+O(\varepsilon_\delta^m)\,.$$

Proof. The unimodality of R_δ, the monotonicity of Φ', and (20) imply

$$0\le\tilde{H}_\delta(\tilde{x})\le\tilde{G}_\delta(\tilde{x})\le C\varepsilon_\delta^{m+1}$$

for all $\tilde{x}\in J_\delta$. Combining this with (44) and Theorem 7 we arrive at the desired estimates for the second derivatives. We also notice that Lemma 6 along with Lemma 14 imply

$$\begin{aligned}\tilde{T}_0'\left(\frac{1}{2\mu_\delta}\right)&=\tfrac{1}{2}\tilde{S}_0'\left(\frac{1}{2\mu_\delta}\right)+\tfrac{1}{2}\overline{\mu}=\overline{\mu}+O(\varepsilon_\delta^m)\\&=\frac{\mu_\delta}{\varepsilon_\delta}R_\delta'\big(\tfrac{1}{2}\big)+O(\varepsilon_\delta^m)=\tilde{T}_\delta'\left(\frac{1}{2\mu_\delta}\right)+O(\varepsilon_\delta^m)\,,\end{aligned}$$

where the last identity stems from (43), and by similar arguments we justify

$$\begin{aligned}\tilde{T}_0\left(\frac{1}{2\mu_\delta}\right)&=\frac{1}{2}\tilde{S}_0\left(\frac{1}{2\mu_\delta}\right)+\frac{1}{4}\frac{\overline{\mu}}{\mu_\delta}+\frac{1}{2}\overline{\kappa}\\&=\frac{1}{2}\left(\frac{\overline{\mu}}{2\mu_\delta}-\overline{\kappa}\right)+\frac{1}{4}\frac{\overline{\mu}}{\mu_\delta}+\frac{1}{2}\overline{\kappa}+O\big(\varepsilon_\delta^{m-1}\big)\\&=\frac{\overline{\mu}}{2\mu_\delta}+O\big(\varepsilon_\delta^{m-1}\big)\\&=\frac{R_\delta\big(\pm\frac{1}{2}\big)}{\varepsilon_\delta}+O\big(\varepsilon_\delta^{m-1}\big)=\tilde{T}_\delta\left(\frac{1}{2\mu_\delta}\right)+O\big(\varepsilon_\delta^{m-1}\big)\,.\end{aligned}$$

The assertions for the first and zeroth derivatives can thus be derived from the estimates for the second derivatives by integration with respect to $\tilde{x}$. □

3.5. Summary on the asymptotic analysis

We finally combine all partial results as follows.

Theorem 16 (global approximation in the high-energy limit)**.** *The formulas*

$$\hat{R}_\varepsilon(x):=\begin{cases}1-\varepsilon-\varepsilon\,\tilde{S}_0\left(\dfrac{|x|}{\hat{\mu}_\varepsilon}\right) & \textit{for } 0\le|x|<\frac{1}{2}\\ \varepsilon\,\tilde{T}_0\left(\dfrac{1-|x|}{\hat{\mu}_\varepsilon}\right) & \textit{for } \frac{1}{2}\le|x|<\frac{3}{2}\\ 0 & \textit{else}\end{cases}$$

and

$$\hat{V}_\varepsilon(x):=\frac{\varepsilon}{\hat{\mu}_\varepsilon}\begin{cases}\tilde{W}_0\left(\dfrac{\frac{1}{2}-|x|}{\hat{\mu}_\varepsilon}\right) & \textit{for } 0\le|x|<1\\ 0 & \textit{else}\end{cases}$$

with

$$\hat{\mu}_\varepsilon := \frac{\overline{\mu}\,\varepsilon}{1+\varepsilon\,(\overline{\kappa}-1)} \qquad \hat{\sigma}_\varepsilon := \varepsilon^{-m-2}\,\hat{\mu}_\varepsilon^2$$

approximate the solitary waves from Assumption 1 in the sense of

$$\|R_\delta - \hat{R}_{\varepsilon_\delta}\|_q + \|V_\delta - \hat{V}_{\varepsilon_\delta}\|_q + \varepsilon_\delta^m|\sigma_\delta - \hat{\sigma}_{\varepsilon_\delta}| = O(\varepsilon_\delta^m) = O(\delta^m)$$

for any $q \in [1, \infty]$. Here, $\tilde{S}_0$ solves the ODE initial value problem (5), the constants $\overline{\mu}$, $\overline{\kappa}$ are given in (24), and the functions $\tilde{W}_0$, $\tilde{T}_0$ are defined in (34), (45).

Proof. Notice that $O(\delta) = O(\varepsilon_\delta)$ is ensured by Lemma 12 and that it suffices to consider the case $q = \infty$ because the functions $\hat{R}_\varepsilon$ and $\hat{V}_\varepsilon$ are compactly supported. Theorems 7, 9, and 15 – which concern the convergence under the different rescalings — as well as the tail estimates from Theorem 10 provide a variant of the desired estimates in which $\hat{\mu}_{\varepsilon_\delta}$ is replaced by μ_δ. Thanks to Lemma 11 we also have $\mu_\delta \sim \delta$ as well as

$$\mu_\delta = \hat{\mu}_{\varepsilon_\delta} + O\big(\delta^{m+1}\big) \qquad \text{and hence} \qquad \frac{1}{\mu_\delta} = \frac{1}{\hat{\mu}_{\varepsilon_\delta}} + O\big(\delta^{m-1}\big). \tag{46}$$

Since $\tilde{S}_0'$ is bounded, the intermediate value theorem implies

$$\varepsilon_\delta\left|\tilde{S}_0\left(\frac{|x|}{\mu_\delta}\right) - \tilde{S}_0\left(\frac{|x|}{\hat{\mu}_{\varepsilon_\delta}}\right)\right| = \varepsilon_\delta\left|\|\tilde{S}_0'\|_\infty\;|x|\;O\big(\delta^{m-1}\big)\right| = O(\delta^m)\,,$$

and by similar arguments we derive the corresponding estimate for $\tilde{T}_0$. For the approximation of the velocity profile, the crucial estimate is

$$\left|\tilde{W}_0\left(\frac{\frac{1}{2}-|x|}{\mu_\delta}\right) - \tilde{W}_0\left(\frac{\frac{1}{2}-|x|}{\hat{\mu}_{\varepsilon_\delta}}\right)\right| = \left|\tilde{W}_0'(\xi)\big(\tfrac{1}{2}-|x|\big)O\big(\delta^{m-1}\big)\right| = O(\delta^m),$$

where ξ denotes an intermediate value and where we used that the function $\tilde{x} \mapsto \tilde{x}\tilde{W}_0'(\tilde{x})$ is bounded. Finally, the estimates for $\sigma_\delta - \hat{\sigma}_{\varepsilon_\delta}$ follow from (4) and (46). □

For practical purposes it might be more convenient to regard ε as the independent parameter and δ as the derived quantity. In this case we can employ the following result, which is, however, weaker than Theorem 16 since the guaranteed error bounds are of lower order.

Corollary 17 (variant of the global approximation result)**.** *We have*

$$\|\hat{R}_\varepsilon - R_{\hat{\delta}_\varepsilon}\|_q + \|\hat{V}_\varepsilon - V_{\hat{\delta}_\varepsilon}\|_q + \varepsilon^{m-1}|\hat{\sigma}_\varepsilon - \sigma_{\hat{\delta}_\varepsilon}| = O\big(\varepsilon^{m-1}\big)$$

for any $q \in [1, \infty]$, where $\hat{\delta}_\varepsilon := 1 - \|\hat{V}_\varepsilon\|_2 \sim \varepsilon$.

Proof. Let ε_* be fixed, where the subscript $*$ has been introduced for the sake of clarity only, and write $\delta_* := \hat{\delta}_{\varepsilon_*}$ as well as $\mu_* := \hat{\mu}_{\varepsilon_*}$ for the quantities that can be computed directly and explicitly from ε_*. Lemma 12 provides

$$(1-\delta_*)^2 = \|V_{\delta_*}\|_2^2 = \Big(1+\varepsilon_{\delta_*}(\overline{\kappa}-1)\Big)\left(1-\varepsilon_{\delta_*}\left(1+\frac{\overline{\eta}}{\overline{\mu}}\right)\right) + O\big(\varepsilon_{\delta_*}^m\big)\,,$$

where ε_{δ_*} and μ_{δ_*} are defined by the exact wave data $(R_{\delta_*},\, V_{\delta_*},\, \sigma_{\delta_*})$ and must not be confused with ε_* and μ_*. On the other hand, a direct calculation – we just repeat all arguments between (41) and (42) with $(\varepsilon_*,\, \mu_*)$ instead of $(\varepsilon_{\delta_*},\, \mu_{\delta_*})$ – reveals

$$\begin{aligned}(1-\delta_*)^2 = \|\hat{V}_{\varepsilon_*}\|_2^2 &= 2\,\frac{\varepsilon_*^2}{\mu_*}\int\limits_{-1/(2\mu_*)}^{+1/(2\mu_*)} \tilde{W}_0(\tilde{x})^2\,\mathrm{d}\tilde{x}\\ &= \Big(1+\varepsilon_*(\overline{\kappa}-1)\Big)\Big(1-\varepsilon_*\Big(1+\frac{\overline{\eta}}{\overline{\mu}}\Big)\Big) + O(\varepsilon_*^m)\,.\end{aligned}$$

Equating the right hand sides in both identities we then conclude

$$\varepsilon_{\delta_*} = \varepsilon_* + O(\varepsilon_*^m)\,,\qquad \mu_{\delta_*} = \mu_* + O(\varepsilon_*^m)\,,$$

where the last identity holds due to the ε_*-dependence of μ_* and Lemma 11, which provides an approximation of μ_{δ_*} in terms of ε_{δ_*}. Finally, exploiting the properties of $\tilde{S}_0$, $\tilde{W}_0$, and $\tilde{T}_0$ as in the proof of Theorem 16 we arrive at

$$\|\hat{R}_{\varepsilon_*} - \hat{R}_{\varepsilon_{\delta_*}}\|_\infty + \|\hat{V}_{\varepsilon_*} - \hat{V}_{\varepsilon_{\delta_*}}\|_\infty = O\Big(\frac{|\varepsilon_* - \varepsilon_{\delta_*}|}{\varepsilon_*}\Big) = O\big(\varepsilon_*^{m-1}\big)\,,$$

and obtain analogous estimates for the q-norms due to the compactness of the supports. The assertion is now provided by Theorem 16. □

3.6. On the asymptotic eigenvalue problem

Of particular interest in the analysis of solitary waves is the spectrum of the linearized equation. The problem consists of finding eigenpairs $(\lambda,\, U) \in \mathbb{R}\times \mathsf{L}^2(\mathbb{R})$ such that

$$\lambda U = L_\delta U\,,\qquad L_\delta U := AQ_\delta AU\,,\qquad Q_\delta(x) := \frac{\Phi''(R_\delta(x))}{\sigma_\delta}\,, \tag{47}$$

where the function Q_δ becomes singular in the limit $\delta \to 0$, see Figure 8. Due to the shift symmetry of the traveling wave equation (6), there is always the solution

$$\lambda = 1\,,\qquad U = V_\delta'\,,$$

and a natural question is whether this eigenspace is simple or not. In fact, simplicity would immediately imply some local uniqueness for solitary waves and is also an important ingredient for both linearized and orbital stability.

Unfortunately, very little is known about the solution set of (47) due to the nonlocality of the operator A. In the small-energy limit of FPU-type chains, the corresponding problem has been solved in [4] by showing that the spectral properties of the analogue to L_δ are governed by an asymptotic ODE problem which stems from the KdV equation and admits explicit solutions. The hope is that the asymptotic formulas derived in this paper provide spectral control in the high-energy limit. A detailed study of the singular perturbation problem (47) is beyond the scope of this paper but preliminary investigations indicate that the spectrum of L_δ depends in the limit $\delta \to 0$ – and at least for sufficiently large m – crucially on the coefficients $c_{\pm 1}$ that are derived in the following result.

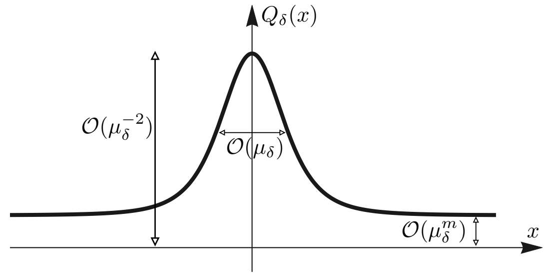

Figure 8. Cartoon of the coefficient function Q_δ in the linear eigenproblem (47).

Theorem 18 (weak⋆-expansion of Q_δ)**.** *For any sufficiently regular test function* φ *we have*

$$\int_{\mathbb{R}} Q_\delta(x)\varphi(x)\,\mathrm{d}x = c_{-1}\mu_\delta^{-1}\varphi(0) + c_{+1}\mu_\delta^{+1}\varphi''(0) + O\Big(\mu_\delta^{\min\{m-2,3\}}\Big)$$

with

$$c_{-1} := \int_{\mathbb{R}} \frac{1}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x}\,, \qquad c_{+1} := \tfrac{1}{2}\int_{\mathbb{R}} \frac{\tilde{x}^2}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x}\,,$$

where the error terms depend on φ*.*

Proof. Due to $\Phi''(r) = (1-r)^{-m-2}$ and the scaling relations (32) we find

$$\int_{\mathbb{R}\setminus[-1/2,+1/2]} Q_\delta(x)\varphi(x)\,\mathrm{d}x = O\big(\sigma_\delta^{-1}\big) = O(\mu_\delta^m)\,,$$

and (15) along with (4) implies

$$\begin{aligned}\int_{[-1/2,+1/2]} Q_\delta(x)\varphi(x)\,\mathrm{d}x &= \frac{\mu_\delta}{\sigma_\delta}\int_{J_\delta} \Phi''(R_\delta(\mu_\delta\tilde{x}))\varphi(\mu_\delta\tilde{x})\,\mathrm{d}\tilde{x} \\ &= \frac{1}{\mu_\delta}\int_{J_\delta} \frac{\varphi(\mu_\delta\tilde{x})}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x}\,.\end{aligned}$$

Theorem 7 as well as the linear growth of $\tilde{S}_0$ – see Lemma 6 – ensure

$$\begin{aligned}\int_{J_\delta} \frac{\varphi(\mu_\delta\tilde{x})}{\Big(1+\tilde{S}_\delta(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x} &= \int_{J_\delta} \frac{\varphi(\mu_\delta\tilde{x})}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x} + O\big(\mu_\delta^{m-1}\big) \\ &= \int_{\mathbb{R}} \frac{\varphi(\mu_\delta\tilde{x})}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x} + O\big(\mu_\delta^{m-1}\big)\end{aligned} \qquad (48)$$

and by smoothness of φ and evenness of $\tilde{S}_0$ we can approximate

$$\int_{\mathbb{R}} \frac{\varphi(\mu_\delta\tilde{x})}{\Big(1+\tilde{S}_0(\tilde{x})\Big)^{m+2}}\,\mathrm{d}\tilde{x} = c_{-1}\varphi(0) + c_{+1}\mu_\delta^2\varphi''(0) + O\big(\mu_\delta^4\big)\,.$$

The claim now follows by combining all partial estimates from above. □

For completeness we mention that the estimate (48) is not optimal for moderate values of m and might be improved for the prize of more technical effort.

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