Rigorous Continuum Limit for the Discrete Network Formation Problem

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Abstract. Motivated by recent physics papers describing the formation of biological transport networks we study a discrete model proposed by Hu and Cai consisting of an energy consumption function constrained by a linear system on a graph. For the spatially two-dimensional rectangular setting we prove the rigorous continuum limit of the constrained energy functional as the number of nodes of the underlying graph tends to infinity and the edge lengths shrink to zero uniformly. The proof is based on reformulating the discrete energy functional as a sequence of integral functionals and proving their Γ-convergence towards the respective continuum energy functional.

Key words: Network formation; Γ-convergence; Continuum limit; Finite Element Discretization.

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Contents

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TABLE 1. Notation. (*) denotes variables that are given as data.

Variable	Meaning	Related to
S_j $(*)$	intensity of source/sink	vertex $j \in \mathbb{V}$
P_j	pressure	vertex $j \in \mathbb{V}$
L_{ij} $(*)$	length of an edge	edge $(i,j) \in \mathbb{E}$
Q_{ij}	flow from $j \in \mathbb{V}$ to $i \in \mathbb{V}$	edge $(i,j) \in \mathbb{E}$
C_{ij}	conductivity	edge $(i,j) \in \mathbb{E}$

1. Introduction

In this paper we derive the rigorous continuum limit of the discrete network formation model of Hu and Cai [13]. The model is posed on an a priori given graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$, consisting of the set of vertices (nodes) $\mathbb{V}$ and the set of unoriented edges (vessels) $\mathbb{E}$. Any pair of vertices i, $j \in \mathbb{V}$ is connected by at most one edge $(i,j) \in \mathbb{E}$, such that the corresponding graph $(\mathbb{V}, \mathbb{E})$ is connected. The lengths $L_{ij} > 0$ of the vessels $(i,j) \in \mathbb{E}$ are given a priori and fixed. The adjacency matrix of the graph $(\mathbb{V}, \mathbb{E})$ is denoted by $\mathbb{A}$, i.e., $\mathbb{A}_{ij} = 1$ if $(i,j) \in \mathbb{E}$, otherwise $\mathbb{A}_{ij} = 0$.

Let us emphasize that by fixing $(\mathbb{V}, \mathbb{E})$, the set of possible flow directions in the network is also fixed. For each node $j \in \mathbb{V}$ we prescribe the strength of source/sink $S_j \in \mathbb{R}$ and we adopt the convention that $S_j > 0$ denotes sources, while $S_j < 0$ sinks. We also allow for $S_j = 0$, i.e., no external in- or outgoing flux in this node. We impose the global mass conservation

$$\sum_{j \in \mathbb{V}} S_j = 0. \tag{1.1}$$

We denote C_{ij} and, resp., Q_{ij} the conductivity and, resp., the flow through the vessel $(i,j) \in \mathbb{E}$. Note that the flow is oriented and we adopt the convention that $Q_{ij} > 0$ means net flow from node $j \in \mathbb{V}$ to node $i \in \mathbb{V}$. An overview of the notation is provided in Table 1. We assume low Reynolds number of the flow through the network, so that the flow rate through a vessel $(i,j) \in \mathbb{E}$ is proportional to its conductivity and the pressure drop between its two ends, i.e.,

$$Q_{ij} = C_{ij} \frac{P_j - P_i}{L_{ij}}. \tag{1.2}$$

Local conservation of mass is expressed in terms of the Kirchhoff law,

$$\sum_{i \in \mathbb{V}} \mathbb{A}_{ij} C_{ij} \frac{P_j - P_i}{L_{ij}} = S_j \qquad \text{for all } j \in \mathbb{V}. \tag{1.3}$$

Note that for any given vector of conductivities $C := (C_{ij})_{(i,j) \in \mathbb{E}}$, (1.3) represents a linear system of equations for the vector of pressures $(P_j)_{j \in \mathbb{V}}$. The system has a solution, unique up to an additive constant, if and only if the graph with edge weights given by C is connected [7], where only edges with positive conductivities are taken into account (i.e., edges with zero conductivity are discarded).

Assuming that the material cost for an edge $(i,j) \in \mathbb{E}$ of the network is proportional to a power C_{ij}^{γ} of its conductivity, Hu and Cai [13] consider the energy consumption function of the form

$$E[C] := \frac{1}{2} \sum_{i \in \mathbb{V}} \sum_{j \in \mathbb{V}} \left(\frac{Q_{ij}^2}{C_{ij}} + \frac{\nu}{\gamma} C_{ij}^{\gamma} \right) \mathbb{A}_{ij} L_{ij}, \tag{1.4}$$

where $\nu > 0$ is the metabolic coefficient and Q_{ij} is given by (1.2), where the pressure drop $\frac{P_j - P_i}{L_{ij}}$ is determined by (1.3). The first part of the energy consumption (1.4) represents the kinetic energy

(pumping power) of the material flow through the vessels, and we shall call it *pumping term* in the sequel. The second part represents the metabolic cost of maintaining the network and shall be called *metabolic term*. For instance, the metabolic cost for a blood vessel is proportional to its cross-section area [14]. Modeling blood flow by Hagen-Poiseuille's law, the conductivity of the vessel is proportional to the square of its cross-section area. This implies $\gamma = 1/2$ for blood vessel systems. For leaf venations, the material cost is proportional to the number of small tubes, which is proportional to C_{ij}, and the metabolic cost is due to the effective loss of the photosynthetic power at the area of the venation cells, which is proportional to $C_{ij}^{1/2}$. Consequently, the effective value of γ typically used in models of leaf venation lies between $1/2$ and 1, [13]. Hu and Cai showed that the optimal networks corresponding to minimizers of (1.3)-(1.4) exhibit a phase transition at $\gamma = 1$, with a "uniform sheet" (the network is tiled with loops) for $\gamma > 1$ and a "loopless tree" for $\gamma < 1$, see also [11]. Moreover, they consider the gradient flow of the energy (1.4) constrained by the Kirchhoff law (1.3), which leads to the ODE system for the conductivities C_{ij},

$$\frac{\mathrm{d}C_{ij}}{\mathrm{d}t} = \left(\frac{Q_{ij}^2}{C_{ij}^2} - \nu C_{ij}^{\gamma-1}\right) L_{ij} \qquad \text{for } (i,j) \in \mathbb{E},$$

coupled to the Kirchhoff law (1.3) through (1.2). This system represents an adaptation model which dynamically responds to local information and can naturally incorporate fluctuations in the flow.

This paper focuses on deriving the rigorous continuum limit of the energy functional (1.3)-(1.4) as the number of nodes of the underlying graph tends to infinity and the edge lengths L_{ij} tend uniformly to zero. In a general setting with a sequence of unstructured graphs this is a mathematically very challenging task. In particular, one has to expect that the object obtained in the limit will depend on the structural and statistical properties of the graph sequence (connectivity, edge directions and density etc.). Therefore, we consider the particular setting where the graphs correspond to regular equidistant meshes in 1D and 2D. As we explain in Section 3, the energy minimization problem for (1.3)-(1.4) in the one-dimensional case is in fact trivial, and the form of the limiting functional is obvious. However, we use this setting as a toy example and carry out the rigorous limit passage anyway. The reason is that in the 1D setting we avoid most of the technical peculiarities of the two-dimensional case and we can focus on the essential idea of the method. Equipped with this insight, we shall turn to the two-dimensional case (Section 4), where the graph is an equidistant rectangular mesh on a square-shaped domain Ω.

In both the 1D and 2D cases, it is necessary to adopt the additional assumption that the conductivities are a priori bounded away from zero. In particular, we introduce a modification of the system (1.3)-(1.4) where the conductivities are of the form $r + C_{ij}$, where $r > 0$ is a fixed global constant. The reason is that we need to guarantee the solvability of the Poisson equation (1.9) below, which will be obtained in the continuum limit. Moreover, in the 2D case, the additive terms in the energy functional have to be scaled by the square of the edge length L_{ij}. This is due to the fact that we are embedding the inherently one-dimensional edges of the graph into two spatial dimensions; see [8, Section 3.2] for details. Thus, we shall work with the energy functional

$$E[C] := \frac{1}{2} \sum_{i \in \mathbb{V}} \sum_{j \in \mathbb{V}} \left(\frac{Q_{ij}^2}{r + C_{ij}} + \frac{\nu}{\gamma} (r + C_{ij})^\gamma \right) \mathbb{A}_{ij} L_{ij}^d, \tag{1.5}$$

where $d = 1$, 2 is the space dimension, coupled to the (properly rescaled) Kirchhoff law

$$\sum_{i \in \mathbb{V}} \mathbb{A}_{ij} (r + C_{ij}) \frac{P_j - P_i}{L_{ij}} = L_j S_j \qquad \text{for all } j \in \mathbb{V} \tag{1.6}$$

through

$$Q_{ij} = (r + C_{ij})\frac{P_j - P_i}{L_{ij}}, \tag{1.7}$$

where L_j are (abstract) weights that scale linearly with the mean edge length; see [8, Section 3.1] for details about the scaling in (1.6). The main benefit of this paper is the rigorous derivation of the limiting energy functional, which for the two-dimensional case is of the form

$$\mathcal{E}[c] = \int_\Omega \nabla p[c] \cdot (rI + c)\nabla p[c] + \frac{\nu}{\gamma}\left(|r + c_1|^\gamma + |r + c_2|^\gamma\right) \mathrm{d}\mathbf{x}, \tag{1.8}$$

with $\mathbf{x} = (x, y) \in \mathbb{R}^2$ and where $p[c] \in H^1(\Omega)$ is a weak solution of the Poisson equation

$$-\nabla \cdot ((rI + c)\nabla p) = S \tag{1.9}$$

subject to no-flux boundary conditions on $\partial\Omega$, where I is the unit matrix and c is the diagonal 2×2-tensor

$$c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{1.10}$$

Here, $S \in L^2(\Omega)$ denotes the source/sink term and in analogy to (1.1) we require $\int_\Omega S \,\mathrm{d}\mathbf{x} = 0$. The derivation is based on three steps:

(1) Establish a connection between the discrete solutions of the Kirchhoff law (1.6) and weak solutions of the Poisson equation (1.9); see Section 3.1 in 1D and Sections 4.1, 4.2 in 2D.
(2) Reformulate the discrete energy functional (1.5) as an integral functional defined on the set of bounded functions; see Section 3.1 in 1D and Section 4.2 in 2D.
(3) Show that the sequence of integral functionals Γ-converges to the energy functional (1.8); see Section 3.2 in 1D and Section 4.3 in 2D. See, e.g., [6, 2] for details about Γ-convergence.

The Γ-convergence opens the door for constructing global minimizers of (1.8)–(1.9) as limits of sequences of minimizers of the discrete problem (1.5)–(1.6). However, for this we need strong convergence of the minimizers in an appropriate topology. In agreement with [9, 10, 3] we introduce diffusive terms into the discrete energy functionals, modeling random fluctuations in the medium (Section 3.3 for 1D and Section 4.4 in 2D). The diffusive terms provide compactness of the minimizing sequences in a suitable topology and facilitate the construction of global minimizers of (1.8)–(1.9).

Let us note that the steepest descent minimization procedure for (1.8)–(1.9) is represented by the formal L^2-gradient flow. This leads to the system of partial differential equations for $c_1 = c_1(t, x, y)$, $c_2 = c_2(t, x, y)$,

$$\begin{aligned} \partial_t c_1 &= (\partial_x p)^2 - \nu(r + c_1)^{\gamma-1}, \\ \partial_t c_2 &= (\partial_y p)^2 - \nu(r + c_2)^{\gamma-1}, \end{aligned} \tag{1.11}$$

subject to homogeneous Dirichlet boundary data and coupled to (1.9) through (1.10). The existence of weak solutions and their properties are studied in [8]. Finally, let us remark that [12] proposed a different PDE system, derived from the discrete model [13] by certain phenomenological considerations (laws of porous medium flow, see [3] for details). The system consists of a parabolic reaction-diffusion equation for the vector-valued conductivity field, constrained by a Poisson equation for the pressure, and was studied in the series of papers [9, 10, 1, 3]. However, a rigorous derivation of the model is still lacking; moreover, no explicit connection to the system (1.11) has been established so far.

2. An auxiliary Lemma

Lemma 1. *Fix $r > 0$, a bounded domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, and $S \in L^2(\Omega)$. Let $(c^N)_{N\in\mathbb{N}} \subset L^\infty(\Omega)$ be a sequence of nonnegative, essentially bounded functions on Ω, such that $c^N \to c \in L^2(\Omega)$ in the norm topology of $L^2(\Omega)$. Let $(p^N)_{N\in\mathbb{N}} \subset H^1(\Omega)$ be a sequence of zero-average weak solutions of the Poisson equation*

$$-\nabla \cdot ((r + c^N)\nabla p^N) = S \tag{2.1}$$

subject to homogeneous Neumann boundary conditions on $\partial\Omega$. Then ∇p^N converges to ∇p and $\sqrt{c^N}\nabla p^N$ converges to $\sqrt{c}\nabla p$ strongly in $L^2(\Omega)$, where p is the zero-average weak solution of

$$-\nabla \cdot ((r + c)\nabla p) = S \tag{2.2}$$

subject to homogeneous Neumann boundary conditions on $\partial\Omega$. In particular, we have

$$\lim_{N\to\infty} \int_\Omega (r + c^N)|\nabla p^N|^2 \,\mathrm{d}\mathbf{x} = \int_\Omega (r + c)|\nabla p|^2 \,\mathrm{d}\mathbf{x}. \tag{2.3}$$

Remark 1. *Note that we do* not *assume that $(c^N)_{N\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\Omega)$, nor that $c \in L^\infty(\Omega)$.*

Proof. Using p^N as a test function in (2.1), due to the nonnegativity of c^N, we have

$$\begin{aligned} r \left\|\nabla p^N\right\|^2_{L^2(\Omega)} \leq \int_\Omega (r + c^N)|\nabla p^N|^2 \,\mathrm{d}\mathbf{x} &= \int_\Omega S p^N \,\mathrm{d}\mathbf{x} \\ &\leq \frac{1}{2\varepsilon} \|S\|^2_{L^2(\Omega)} + \frac{\varepsilon C_P}{2} \left\|\nabla p^N\right\|^2_{L^2(\Omega)}, \end{aligned} \tag{2.4}$$

where C_P is the Poincaré constant. With a suitable choice of $\varepsilon > 0$ we obtain a uniform estimate on p^N in $H^1(\Omega)$. Consequently, there exists a subsequence of p^N that converges weakly in $H^1(\Omega)$ to some $p \in H^1(\Omega)$. Since $c^N \to c$ strongly in $L^2(\Omega)$, we can pass to the limit in the distributional formulation of (2.1) to obtain

$$\int_\Omega (r + c)\nabla p \cdot \nabla \phi \,\mathrm{d}\mathbf{x} = \int_\Omega S\phi \,\mathrm{d}\mathbf{x} \qquad \text{for all } \phi \in C_0^\infty(\Omega). \tag{2.5}$$

Noting that (2.4) also implies a uniform bound on $\int_\Omega c^N|\nabla p^N|^2 \,\mathrm{d}\mathbf{x}$, we have due to the weak lower semicontinuity of the L^2-norm,

$$\int_\Omega (r + c)|\nabla p|^2 \,\mathrm{d}\mathbf{x} \leq \liminf_{N\to\infty} \int_\Omega (r + c^N)|\nabla p^N|^2 \,\mathrm{d}\mathbf{x} < +\infty. \tag{2.6}$$

Consequently, we can use p as a test function in (2.5) to obtain

$$\int_\Omega (r + c)|\nabla p|^2 \,\mathrm{d}\mathbf{x} = \int_\Omega Sp \,\mathrm{d}\mathbf{x}.$$

Therefore, using p^N as a test function in (2.1),

$$\lim_{N\to\infty} \int_\Omega (r + c^N)|\nabla p^N|^2 \,\mathrm{d}\mathbf{x} = \lim_{N\to\infty} \int_\Omega Sp^N \,\mathrm{d}\mathbf{x} = \int_\Omega Sp \,\mathrm{d}\mathbf{x} = \int_\Omega (r + c)|\nabla p|^2 \,\mathrm{d}\mathbf{x},$$

which gives (2.3) and, further,

$$\begin{aligned} \limsup_{N\to\infty} \int_\Omega |\nabla p^N|^2 \,\mathrm{d}\mathbf{x} &\leq \limsup_{N\to\infty} \int_\Omega (r + c^N)|\nabla p^N|^2 \,\mathrm{d}\mathbf{x} + \limsup_{N\to\infty} \left(- \int_\Omega c^N |\nabla p^N|^2 \,\mathrm{d}\mathbf{x} \right) \\ &= \int_\Omega (r + c)|\nabla p|^2 \,\mathrm{d}\mathbf{x} - \liminf_{N\to\infty} \int_\Omega c^N |\nabla p^N|^2 \,\mathrm{d}\mathbf{x}. \end{aligned}$$

Now, using (2.6), we have

$$-\liminf_{N\to\infty}\int_\Omega c^N|\nabla p^N|^2\,\mathrm{d}\mathbf{x} = -\liminf_{N\to\infty}\int_\Omega |\sqrt{c^N}\nabla p^N|^2\,\mathrm{d}\mathbf{x} \le -\int_\Omega |\sqrt{c}\nabla p|^2\,\mathrm{d}\mathbf{x}.$$

Therefore,

$$\limsup_{N\to\infty}\int_\Omega |\nabla p^N|^2\,\mathrm{d}\mathbf{x} \le \int_\Omega |\nabla p|^2\,\mathrm{d}\mathbf{x},$$

so that $\lim_{N\to\infty}\left\|\nabla p^N\right\|_{L^2(\Omega)} = \|\nabla p\|_{L^2(\Omega)}$, which directly implies that (a subsequence of) p^N converges towards p strongly in $H^1(\Omega)$.

■

3. The 1D equidistant setting

In this section we consider the spatially one-dimensional setting of the discrete network formation problem, where the graph $(\mathbb{V},\mathbb{E})$ is given as a mesh on the interval $[0,1]$. Moreover, for simplicity we consider the equidistant case, where for a fixed $N\in\mathbb{N}$ construct the sequence of meshpoints x_i,

$$x_i = ih \qquad \text{for } i=0,\dots,N, \text{ with } h := 1/N.$$

We identify the meshpoints x_i with the vertices of the graph, i.e., we set $\mathbb{V} := \{x_i;\ i=0,\dots,N\}$. The segments (x_{i-1},x_i) connecting any two neighboring nodes are identified with the edges of the graph, i.e., $\mathbb{E} := \{(x_{i-1},x_i);\ i=1,\dots,N\}$. By a slight abuse of notation, we shall write $i\in\mathbb{V}$ instead of $x_i\in\mathbb{V}$ in the sequel, and similarly $i\in\mathbb{E}$ instead of $(x_{i-1},x_i)\in\mathbb{E}$. Moreover, we shall use the notation $C := (C_i)_{i=1}^N$ with $C_i\ge 0$ the conductivity of the edge $i\in\mathbb{E}$, $P_i\in\mathbb{R}$ for the pressure in node $i\in\mathbb{V}$ and $S_i^N\in\mathbb{R}$ for the source/sink in node $i\in\mathbb{V}$ with $\sum_{i=1}^N S_i^N = 0$ by (1.1). With this notation we rewrite the energy functional (1.5) as $E^N[C]:\mathbb{R}_+^N\mapsto\mathbb{R}$,

$$E^N[C] := h\sum_{i=1}^N \frac{Q_i^2}{r+C_i} + \frac{\nu}{\gamma}(r+C_i)^\gamma, \tag{3.1}$$

with the fluxes

$$Q_i := (r+C_i)\frac{P_{i-1}-P_i}{h}, \qquad \text{for } i=1,\dots,N. \tag{3.2}$$

Note that we orient the fluxes Q_i such that $Q_i>0$ if the material flows from x_{i-1} to x_i. The Kirchhoff law (1.6) is then written in the form

$$(r+C_i)\frac{P_i-P_{i-1}}{h} + (r+C_{i+1})\frac{P_i-P_{i+1}}{h} = hS_i^N \qquad \text{for } i=1,\dots,N-1, \tag{3.3}$$

while for the terminal nodes we have

$$(r+C_1)\frac{P_0-P_1}{h} = hS_0^N, \qquad (r+C_N)\frac{P_N-P_{N-1}}{h} = hS_N^N\,.$$

Obviously, in the 1D setting the fluxes Q_i are explicitly calculable from the given set of sources/sinks $(S_i)_{i=0}^N$ since the Kirchhoff law (3.3) is the chain of equations

$$\begin{aligned} Q_1 &= hS_0^N, \\ -Q_i+Q_{i+1} &= hS_i^N \quad \text{for } i=1,\dots,N-1, \\ -Q_N &= hS_N^N, \end{aligned}$$

which has the explicit solution

$$Q_i = h\sum_{j=0}^{i-1} S_j^N \qquad \text{for } i = 1, \dots, N-1. \tag{3.4}$$

Note that due to the assumption of the global mass balance (1.1) the "terminal condition" for $i = N$ is implicitly satisfied,

$$-Q_N = -h\sum_{j=0}^{N-1} S_j = hS_N^N. \tag{3.5}$$

With the fluxes given by (3.4)–(3.5), it is trivial to find the global energy minimizer of (3.1), namely, $(r + C_i)^{\gamma+1} = Q_i^2/\nu$. It is also easy to prove that the sequence of the functionals (3.1) converges as $h = 1/N \to 0$ to the continuous functional

$$\mathcal{E}[c] := \int_0^1 \frac{q(x)^2}{r + c(x)} + \frac{\nu}{\gamma}(r + c(x))^\gamma \,\mathrm{d}x, \tag{3.6}$$

with $q(x) := \int_0^x S(\sigma)\,\mathrm{d}\sigma$, in the sense of Riemannian sums if c is a continuous, nonnegative function. Therefore, the limit passage to continuum description in the one-dimensional case is trivial. However, we shall use it as a "training example" which avoids most of the technical difficulties of the two-dimensional setting to gain a clear understanding of the main ideas of the method.

Therefore, we shall ignore the explicit formula (3.4) for the fluxes Q_i and study the limit as $h = 1/N \to 0$ of the sequence of energy functionals (3.1)–(3.2), i.e.,

$$E^N[C] = h\sum_{i=1}^{N}(r + C_i)\left(\frac{P_i - P_{i-1}}{h}\right)^2 + \frac{\nu}{\gamma}(r + C_i)^\gamma, \tag{3.7}$$

where the pressures P_i are calculated as a solution of the Kirchhoff law (3.3). Note that since $r + C_i > 0$ for all $i \in \mathbb{V}$, (3.3) is solvable, uniquely up to an additive constant. In the following we shall show that the sequence (3.7) converges, as $h = \frac{1}{N} \to 0$, to the functional (3.6) with $q := (r + c)\partial_x p[c]$, i.e.,

$$\mathcal{E}[c] = \int_0^1 (r + c)(\partial_x p[c])^2 + \frac{\nu}{\gamma}(r + c)^\gamma \,\mathrm{d}x, \tag{3.8}$$

where $p[c] \in H^1(0, 1)$ is a weak solution of the Poisson equation

$$-\partial_x((r + c)\partial_x p) = S \tag{3.9}$$

on $(0, 1)$, subject to no-flux boundary conditions. Here and in the sequel we fix the source/sink term $S \in L^2(0, 1)$ and, in agreement with (1.1), we assume the global mass balance $\int_0^1 S(x)\,\mathrm{d}x = 0$. Since for $c(x) \geq 0$ the weak solution $p = p(x)$ of (3.9) is unique up to an additive constant, we shall, without loss of generality, always choose the zero-average solution, i.e., $\int_0^1 p(x)\,\mathrm{d}x = 0$.

We shall proceed in several steps: First, we put the discrete energy functionals (3.7) into an integral form, and find an equivalence between solutions of the Kirchhoff law (3.3) and the above Poisson equation with appropriate conductivity. Then we show the convergence of the sequence of reformulated discrete energy functionals towards a continuum one as $h = 1/N \to 0$. Finally, we introduce a diffusive term into the energy functional, which will allow us to construct global minimizers of the continuum energy functional.

3.1. **Reformulation of the discrete energy functional.** In the first step we reformulate the energy functionals (3.7) such that they are defined on the space $L^\infty_+(0,1)$ of essentially bounded nonnegative functions on $(0,1)$. For this purpose, we define the sequence of operators $\mathbb{Q}_0^N : \mathbb{R}^N \to L^\infty(0,1)$ by

$$\mathbb{Q}_0^N : (C_i)_{i=1}^N \mapsto c, \qquad \text{with } c(x) \equiv C_i \text{ for } x \in (x_{i-1}, x_i),\ i = 1, \dots, N.$$

I.e., $\mathbb{Q}_0^N$ maps the sequence $(C_i)_{i=1}^N$ onto the bounded function $c = c(x)$, constant on each interval (x_{i-1}, x_i), $i = 1, \dots, N$. Then, we define the functionals $\mathcal{E}^N : L^\infty_+(0,1) \mapsto \mathbb{R}$,

$$\mathcal{E}^N[c] := \int_0^1 (r + c) \left(\mathbb{Q}_0^N[\Delta^h P] \right)^2 + \frac{\nu}{\gamma}(r + c)^\gamma \, \mathrm{d}x, \tag{3.10}$$

with

$$(\Delta^h P)_i := \frac{P_i - P_{i-1}}{h}, \quad i = 1, \dots, N, \tag{3.11}$$

and $P = (P_i)_{i=0}^N$ a solution of the Kirchhoff law (3.3) with the conductivities $C = (C_i)_{i=1}^N$,

$$C_i := \frac{1}{h} \int_{x_{i-1}}^{x_i} c(x) \, \mathrm{d}x, \quad i = 1, \dots, N.$$

Then, noting that for each $C = (C_i)_{i=1}^N \in \mathbb{R}^N_+$,

$$\frac{1}{h} \int_{x_{i-1}}^{x_i} \mathbb{Q}_0^N[C](x) \, \mathrm{d}x = C_i \qquad \text{for all } i = 1, \dots, N,$$

the discrete energy functional (3.7) can be written in the integral form as $E^N[C] = \mathcal{E}^N[\mathbb{Q}_0^N[C]]$.

Moreover, we establish a connection between the solutions of the Kirchhoff law (3.3) and weak solutions of the Poisson equation (3.9) with $c = \mathbb{Q}_0^N[C]$:

Lemma 2. *For any $C = (C_i)_{i=1}^N \in \mathbb{R}^N_+$ and $S \in L^2(0,1)$ with $\int_0^1 S(x) \, \mathrm{d}x = 0$, let $p = p(x) \in H^1(0,1)$ be a weak solution of the Poisson equation (3.9) with $c = \mathbb{Q}_0^N[C]$, i.e.,*

$$-\partial_x \left((r + \mathbb{Q}_0^N[C]) \partial_x p \right) = S, \tag{3.12}$$

subject to no-flux boundary conditions on $(0,1)$. Then,

$$P_i := p(x_i), \qquad i = 0, \dots, N, \tag{3.13}$$

is a solution of the Kirchhoff law (3.3) with the conductivities $C = (C_i)_{i=1}^N$ and the source/sink terms

$$S_i^N := \frac{1}{h} \int_0^1 S(x) \phi_i^N(x) \, \mathrm{d}x, \qquad i = 0, \dots, N, \tag{3.14}$$

with the hat functions $\phi_i^N = \phi_i^N(x)$ defined in (3.15) below.

Proof. Note that for any $C \in \mathbb{R}^N_+$ there exists a weak solution $p = p(x) \in H^1(0,1)$ of (3.12), unique up to an additive constant. For $i = 1, \dots, N$ we construct the family of piecewise linear test functions ϕ_i^N, supported on (x_{i-1}, x_{i+1}), with

$$\phi_i^N(x) = \begin{cases} 1 + \frac{x - x_i}{h} & \text{for } x \in (x_{i-1}, x_i), \\ 1 - \frac{x - x_i}{h} & \text{for } x \in (x_i, x_{i+1}). \end{cases} \tag{3.15}$$

Using the hat function ϕ_i^N as a test function in (3.12), we obtain

$$(r + C_i) \frac{p(x_i) - p(x_{i-1})}{h} + (r + C_{i+1}) \frac{p(x_i) - p(x_{i+1})}{h} = h S_i^N,$$

where we used the fact that, by construction, $\mathbb{Q}_0^N[C] \equiv C_i$ on the interval (x_{i-1}, x_i). Note that due to the embedding $H^1(0,1) \hookrightarrow C(0,1)$ any weak solution $p = p(x)$ of (3.12) is a continuous function on $[0,1]$, so the pointwise values $p(x_i)$ are well defined for all $i = 0, \dots, N$. Thus, defining P_i as in (3.13) we obtain a solution of the Kirchhoff law (3.3) with the conductivities $C = (C_i)_{i=1}^N$ and source/sink terms (3.14).

■

Note that since $\frac{1}{h}\int_0^1 \phi_i^N(x)\,\mathrm{d}x = 1$ and $S \in L^2(0,1)$, the Lebesgue differentiation theorem gives

$$S_i^N = \frac{1}{h}\int_0^1 S(x)\phi_i^N(x)\,\mathrm{d}x \to S(\overline{x}) \qquad \text{for a.e. } \overline{x} = x_i \text{ as } h = 1/N \to 0.$$

Consequently, for a fixed $S \in L^2(0,1)$ and any $N \in \mathbb{N}$, we have the following reformulation of the discrete problem:

Proposition 1. *For any vector $C = (C_i)_{i=1}^N \in \mathbb{R}_+^N$, we have*

$$E^N[C] = \mathcal{E}^N[\mathbb{Q}_0^N[C]],$$

where $E^N[C]$ is the discrete energy functional (3.7) *coupled to the Kirchhoff law* (3.3) *with sources/sinks S_i^N given by* (3.14)*, and $\mathcal{E}^N$ is the integral form* (3.10)–(3.11) *with the pressures given by $P_i = p(x_i)$, $i = 0, \dots, N$, where $p \in H^1(0,1)$ solves the Poisson equation* (3.12)*.*

3.2. **Convergence of the energy functionals.** Due to Proposition 1, we are motivated to prove the convergence of the sequence of functionals $\mathcal{E}^N$ given by (3.10)–(3.11) towards $\mathcal{E}[c]$ given by (3.8) with $p[c] \in H^1(0,1)$ a weak solution of (3.9) with conductivity $c = c(x)$, equipped with no-flux boundary conditions. We choose to work in the space of essentially bounded functions on $(0,1)$ equipped with the topology of $L^2(0,1)$. The choice of topology is motivated by the need for strong convergence of piecewise constant approximations of bounded functions. Of course, this is true in $L^q(0,1)$ with any $q < +\infty$; our particular choice of $L^2(0,1)$ is further dictated by the fact that we shall apply Lemma 1 in the sequel.

Lemma 3. *Let $\gamma \geq 0$. For any sequence of nonnegative functions $(c^N)_{N\in\mathbb{N}}$, uniformly bounded in $L^\infty(0,1)$ and such that $c^N \to c$ in the norm topology of $L^2(0,1)$ as $N \to \infty$, we have*

$$\mathcal{E}^N[c^N] \to \mathcal{E}[c] \qquad \text{as } h = 1/N \to 0.$$

Proof. By assumption, $c^N \to c$ in the norm topology of $L^2(0,1)$. Consequently, there is a subsequence converging almost everywhere on $(0,1)$ to c, and thus $\left(r + c^N(x)\right)^\gamma$ converges almost everywhere to $(r + c(x))^\gamma$. Since, by assumption, the sequence $\left(r + c^N(x)\right)^\gamma$ is uniformly bounded in $L^\infty(0,1)$, we have by the dominated convergence theorem

$$\int_0^1 \left(r + c^N(x)\right)^\gamma \,\mathrm{d}x \to \int_0^1 (r + c(x))^\gamma \,\mathrm{d}x \qquad \text{as } h = 1/N \to 0.$$

We recall that the pumping part of the discrete energy $\mathcal{E}^N[c^N]$ (3.10) is of the form

$$\int_0^1 (r + c^N)\left(\mathbb{Q}_0^N[\Delta^h p^N]\right)^2 \,\mathrm{d}x, \tag{3.16}$$

with

$$(\Delta^h p^N)_i := \frac{p^N(x_i) - p^N(x_{i-1})}{h}, \quad i = 1, \dots, N,$$

where $p^N \in H^1(0,1)$ is a solution of the Poisson equation (3.9) with conductivity c^N, subject to the no-flux boundary condition. Let us show that (a subsequence of) $\mathbb{Q}_0^N[\Delta^h p^N]$ converges to $\partial_x p[c]$ strongly in $L^2(0,1)$. We proceed in three steps:

- **Weak convergence.** By Jensen inequality we have

$$\begin{aligned} \left\| \mathbb{Q}_0^N[\Delta^h p^N] \right\|^2_{L^2(0,1)} &= h \sum_{i=1}^N \left(\frac{p^N(x_i) - p^N(x_{i-1})}{h} \right)^2 \\ &= h \sum_{i=1}^N \left(\frac{1}{h} \int_{x_{i-1}}^{x_i} \partial_x p^N(x)\,\mathrm{d}x \right)^2 \leq \int_0^1 (\partial_x p^N)^2\,\mathrm{d}x. \end{aligned} \tag{3.17}$$

  Due to the nonnegativity of the functions c^N, the right-hand side is uniformly bounded. Consequently, there exists a weakly converging subsequence of $\mathbb{Q}_0^N[\Delta p^N]$ in $L^2(0,1)$.
- **Identification of the limit.** For a smooth, compactly supported test function $\psi \in C_0^\infty(0,1)$ we write

$$\begin{aligned} \int_0^1 \mathbb{Q}_0^N[\Delta^h p^N](x)\psi(x)\,\mathrm{d}x &= \sum_{i=1}^N \frac{p^N(x_i) - p^N(x_{i-1})}{h} \int_{x_{i-1}}^{x_i} \psi(x)\,\mathrm{d}x \\ &= \frac{1}{h} \sum_{i=1}^{N-1} p^N(x_i) \left(\int_{x_{i-1}}^{x_i} \psi(x)\,\mathrm{d}x - \int_{x_i}^{x_{i+1}} \psi(x)\,\mathrm{d}x \right) + \text{“boundary terms”}, \end{aligned}$$

  where “boundary terms” are the two terms with $i = 0$ and $i = N$, which we however can neglect for large enough N since ψ has a compact support. Then, Taylor expansion for ψ gives

$$\int_{x_{i-1}}^{x_i} \psi(x)\,\mathrm{d}x - \int_{x_i}^{x_{i+1}} \psi(x)\,\mathrm{d}x = -h \int_{x_{i-1}}^{x_i} \partial_x \psi(x)\,\mathrm{d}x + \frac{h^2}{2} \int_{x_{i-1}}^{x_i} \partial^2_{xx} \psi(\xi(x))\,\mathrm{d}x,$$

  with $\xi(x) \in (x_{i-1}, x_i)$. Due to the estimate

$$\left| \frac{h^2}{2} \int_{x_{i-1}}^{x_i} \partial^2_{xx} \psi(\xi(x))\,\mathrm{d}x \right| \leq \frac{h^3}{2} \left\| \partial^2_{xx} \psi \right\|_{L^\infty(0,1)}$$

  we have

$$\int_{x_{i-1}}^{x_i} \psi(x)\,\mathrm{d}x - \int_{x_i}^{x_{i+1}} \psi(x)\,\mathrm{d}x = -h \int_{x_{i-1}}^{x_i} \partial_x \psi(x)\,\mathrm{d}x + \mathcal{O}(h^3),$$

  so that

$$\int_0^1 \mathbb{Q}_0^N[\Delta^h p^N](x)\psi(x)\,\mathrm{d}x = -\int_0^1 \overline{p^N} \partial_x \psi(x)\,\mathrm{d}x + \mathcal{O}(h),$$

  where $\overline{p^N}$ is the piecewise constant function

$$\overline{p^N}(x) \equiv p^N(x_i) \quad \text{for } x \in (x_{i-1}, x_i],\ i = 1, \ldots, N.$$

  It is easy to check that, due to the strong convergence of c^N towards c in $L^2(0,1)$, p^N converges to $p[c]$ weakly in $H^1(0,1)$. Due to the compact embedding $H^1(0,1) \hookrightarrow C(0,1)$,

(a subsequence of) p^N converges uniformly to $p[c]$ on $(0,1)$, and, therefore $\overline{p^N}$ converges strongly to $p[c]$. Therefore,

$$\begin{aligned}\int_0^1 \mathbb{Q}_0^N[\Delta^h p^N](x)\psi(x)\,\mathrm{d}x &\to -\int_0^1 p(x)\partial_x\psi(x)\,\mathrm{d}x \qquad \text{as } h=1/N\to 0,\\ &= \int_0^1 \psi(x)\partial_x p(x)\,\mathrm{d}x.\end{aligned}$$

We conclude that weak limit of (the subsequence of) $\mathbb{Q}_0^N[\Delta^h p^N]$ is $\partial_x p[c]$.

- **Strong convergence.** Finally, due to (3.17), we have

$$\begin{aligned}\left\|\mathbb{Q}_0^N[\Delta^h p^N]-\partial_x p[c]\right\|^2_{L^2(0,1)} &= \left\|\mathbb{Q}_0^N[\Delta^h p^N]\right\|^2_{L^2(0,1)} - 2\langle \mathbb{Q}_0^N[\Delta^h p^N],\partial_x p[c]\rangle_{L^2(0,1)} + \|\partial_x p[c]\|^2_{L^2(0,1)}\\ &\le \left\|\partial_x p^N\right\|^2_{L^2(0,1)} - 2\langle \mathbb{Q}_0^N[\Delta^h p^N],\partial_x p[c]\rangle_{L^2(0,1)} + \|\partial_x p[c]\|^2_{L^2(0,1)},\end{aligned}$$

which vanishes in the limit $h=1/N\to 0$ due to the weak convergence of $\mathbb{Q}_0^N[\Delta p^N]$ and strong convergence of $\partial_x p^N$ in $L^2(0,1)$ due to Lemma 1. Thus, $\mathbb{Q}_0^N[\Delta p^N]$ converges strongly to $\partial_x p[c]$ in $L^2(0,1)$.

We conclude that due to the weak-$*$ convergence of $(r+c^N)$ towards $(r+c)$ in $L^\infty(0,1)$, and strong convergence of $\left(\mathbb{Q}_0^N[\Delta^h p^N]\right)^2$ towards $(\partial_x p[c])^2$ in $L^1(0,1)$, we can pass to the limit as $h=1/N\to 0$ in (3.16) to obtain

$$\int_0^1 (r+c)\left(\partial_x p[c]\right)^2\,\mathrm{d}x.$$

■

3.3. **Introduction of diffusion and construction of continuum energy minimizers.** In Lemma 3 we proved the convergence of the sequence of energy functionals $\mathcal{E}^N$ towards $\mathcal{E}$, i.e., for any $c^N\to c$ in the norm topology of $L^2(0,1)$, we have $\mathcal{E}^N[c^N]\to\mathcal{E}[c]$ as $N\to\infty$. In order to construct energy minimizers of $\mathcal{E}$ as limits of sequences of minimizers of the functionals $\mathcal{E}^N$, we need to introduce a term into $\mathcal{E}^N$ that shall guarantee compactness of the sequence of discrete minimizers. This is done, in agreement with [9, 10, 3], by introducing a diffusive term into the discrete energy functional (3.7), modeling random fluctuations in the medium. Thus, we construct the sequence $E^N_{\mathrm{diff}}:\mathbb{R}^N_+\to\mathbb{R}$,

$$E^N_{\mathrm{diff}}[C] := D^2 h\sum_{i=1}^{N-1}\left(\frac{C_{i+1}-C_i}{h}\right)^2 + E^N[C], \tag{3.18}$$

with $E^N[C]$ defined in (3.7), coupled to the Kirchhoff law (3.3) with sources/sinks S_i^N given by (3.14), and $D^2>0$ the diffusion constant. Note that the new term is a discrete Laplacian acting on the conductivities C.

We now need to reformulate the discrete energy functionals (3.18) in terms of integrals. For this sake, we construct the sequence of operators $\mathbb{Q}_1^N:\mathbb{R}^N\to C(0,1)$, where $\mathbb{Q}_1^N[C]$ is a continuous function on $[0,1]$, linear on each interval $(x_i-h/2,x_i+h/2)$, with

$$\mathbb{Q}_1^N[C](x_i-h/2)=C_i \quad \text{for } i=1,\dots,N,$$

and

$$\mathbb{Q}_1^N[C](x)\equiv C_1 \quad \text{for } x\in[0,h/2), \qquad \mathbb{Q}_1^N[C](x)\equiv C_N \quad \text{for } x\in(1-h/2,1].$$

Then we write the finite difference term in (3.18) as

$$D^2 h \sum_{i=1}^{N-1} \left(\frac{C_{i+1} - C_i}{h} \right)^2 = D^2 \int_0^1 \left(\partial_x \mathbb{Q}_1^N[C] \right)^2 \, \mathrm{d}x,$$

and we have

Proposition 2. *For any vector $C = (C_i)_{i=1}^N \in \mathbb{R}_+^N$,*

$$E_{\mathrm{diff}}^N[C] = D^2 \int_0^1 \left(\partial_x \mathbb{Q}_1^N[C] \right)^2 \, \mathrm{d}x + \mathcal{E}^N[\mathbb{Q}_0^N[C]] \,,$$

where E_{diff}^N defined in (3.18) *and $\mathcal{E}^N$ is given by* (3.10)–(3.11) *with the pressures given by $P_i = p(x_i)$, $i = 0, \dots, N$, where $p \in H^1(0,1)$ solves the Poisson equation* (3.12).

We are now ready to prove the main result of this section:

Theorem 1. *Let $\gamma \geq 0$, $S \in L^2(0,1)$ with $\int_0^1 S(x) \, \mathrm{d}x$ and S_i^N given by* (3.14). *Let $(C^N)_{N \in \mathbb{N}}$ be a sequence of global minimizers of the discrete energy functionals E_{diff}^N given by* (3.18). *Then the sequence $\mathbb{Q}_1^N[C^N]$ converges weakly in $H^1(0,1)$ to $c \in H^1(0,1)$, a global minimizer of the functional $\mathcal{E}_{\mathrm{diff}} : H_+^1(0,1) \to \mathbb{R}$,*

$$\mathcal{E}_{\mathrm{diff}}[c] := D^2 \int_0^1 (\partial_x c)^2 \, \mathrm{d}x + \mathcal{E}[c],$$

where $\mathcal{E}[c]$ is given by (3.8).

Proof. Let us observe that

$$E_{\mathrm{diff}}^N[C^N] \leq E_{\mathrm{diff}}^N[0] = rh \sum_{i=1}^{N} \left(\frac{\widetilde{P}_i - \widetilde{P}_{i-1}}{h} \right)^2 + \frac{\nu}{\gamma} r^\gamma,$$

where $(\widetilde{P}_i)_{i=1}^N$ is a solution of the Kirchhoff law (3.3) with zero conductivities and sources/sinks given by (3.14). Thus, $\widetilde{P}_i = \widetilde{p}(x_i)$ for $i = 1, \dots, N$, where $\widetilde{p} = \widetilde{p}(x)$ is a weak solution of $-r\Delta p = S$ subject to no-flux boundary conditions. Then we have by the Jensen inequality

$$\begin{aligned} D^2 h \sum_{i=1}^{N} \left(\frac{\widetilde{P}_i - \widetilde{P}_{i-1}}{h} \right)^2 &= D^2 h \sum_{i=1}^{N} \left(\frac{1}{h} \int_{x_{i-1}}^{x_i} \partial_x \widetilde{p} \, \mathrm{d}x \right)^2 \\ &\leq D^2 \int_0^1 (\partial_x \widetilde{p})^2 \, \mathrm{d}x. \end{aligned}$$

Consequently, the sequence $\mathcal{E}_{\mathrm{diff}}^N[C^N]$ is uniformly bounded.

Since the sequence

$$D^2 \int_0^1 \left(\partial_x \mathbb{Q}_1^N[C^N] \right)^2 \, \mathrm{d}x = D^2 h \sum_{i=1}^{N-1} \left(\frac{C_{i+1} - C_i}{h} \right)^2 \leq E_{\mathrm{diff}}^N[C^N]$$

is uniformly bounded, there exists a subsequence of $\mathbb{Q}_1^N[C^N]$ converging to some $c \in H^1(0,1)$ weakly in $H^1(0,1)$, and strongly in $L^2(0,1)$; moreover, the sequence is uniformly bounded in $L^\infty(0,1)$. It is easy to check that also $\mathbb{Q}_0^N[C^N]$ converges to c strongly in $L^2(0,1)$, and is uniformly bounded

in $L^\infty(0,1)$. Therefore, by Lemma 3, we have $E^N[C^N] = \mathcal{E}^N[\mathbb{Q}_0^N[C^N]] \to \mathcal{E}[c]$ as $h = 1/N \to 0$. Moreover, due to the weak lower semicontinuity of the L^2-norm, we have

$$\int_0^1 (\partial_x c)^2 \, \mathrm{d}x \le \liminf_{N\to\infty} \int_0^1 \left(\partial_x \mathbb{Q}_1^N[C^N]\right)^2 \, \mathrm{d}x \,.$$

Consequently,

$$\mathcal{E}_{\mathrm{diff}}[c] \le \liminf_{N\to\infty} E^N_{\mathrm{diff}}[C^N]. \tag{3.19}$$

We claim that c is a global minimizer of $\mathcal{E}_{\mathrm{diff}}$ in $H^1_+(0,1)$. For contradiction, assume that there exists $\overline{c} \in H^1_+(0,1)$ such that

$$\mathcal{E}_{\mathrm{diff}}[\overline{c}] < \mathcal{E}_{\mathrm{diff}}[c].$$

We define the sequence $(\overline{C}^N)_{N\in\mathbb{N}}$ by

$$\overline{C}^N_i := \frac{1}{h}\int_{x_{i-1}}^{x_i} \overline{c}(x)\,\mathrm{d}x, \quad i = 1,\dots,N.$$

Then, by assumption, we have for all $N \in \mathbb{N}$,

$$E^N_{\mathrm{diff}}[\overline{C}^N] \ge E^N_{\mathrm{diff}}[C^N]. \tag{3.20}$$

It is easy to check that the sequence $\mathbb{Q}_1^N[\overline{C}^N]$ converges strongly in $H^1(0,1)$ towards $\overline{c}$, therefore

$$\int_0^1 \left(\partial_x \mathbb{Q}_1^N[\overline{C}^N]\right)^2 \mathrm{d}x \to \int_0^1 (\partial_x \overline{c})^2 \, \mathrm{d}x \qquad \text{as } h = 1/N \to 0.$$

Moreover, the sequence $\mathbb{Q}_0^N[\overline{C}^N]$ converges to $\overline{c}$ strongly in $L^2(0,1)$, therefore, by Lemma 3, $\mathcal{E}^N[\mathbb{Q}_0^N[\overline{C}^N]] \to \mathcal{E}[\overline{c}]$ as $h = 1/N \to 0$. Consequently,

$$\lim_{h=1/N\to 0} E^N_{\mathrm{diff}}[\overline{C}^N] = \mathcal{E}_{\mathrm{diff}}[\overline{c}] < \mathcal{E}_{\mathrm{diff}}[c],$$

a contradiction to (3.19)–(3.20).

■

4. The 2D rectangular equidistant setting

In this section we consider the spatially two-dimensional setting of the discrete network formation problem, where the graph $(\mathbb{V},\mathbb{E})$ is embedded in the rectangle $\Omega := [0,1]^2$. We introduce the notation $\mathbf{x} := (x,y) \in \Omega$. For $N \in \mathbb{N}$ we construct the sequence of equidistant rectangular meshes in Ω with mesh size $h := 1/N$ and mesh nodes $\mathbf{X}_i = (X_i, Y_i)$,

$$X_i = (i \bmod N+1)h, \quad Y_i = (i \operatorname{div} N+1)h, \qquad \text{for } i = 0,\dots,(N+1)^2-1, \text{ with } h := 1/N,$$

where $(i \operatorname{div} N+1)$ denotes the integer part of $i/(N+1)$ and $(i \bmod N+1)$ the remainder. We identify the mesh nodes $\mathbf{X}_i = (X_i, Y_i)$ with the vertices of the graph, i.e., we set $\mathbb{V} := \{\mathbf{X}_i;\ i = 0,\dots,(N+1)^2-1\}$. By a slight abuse of notation, we shall write $i \in \mathbb{V}$ instead of $X_i \in \mathbb{V}$ in the sequel. For each node $\mathbf{X}_i$, we denote by $\mathbf{X}_{i,E}$, $\mathbf{X}_{i,W}$, $\mathbf{X}_{i,N}$, $\mathbf{X}_{i,S}$ its direct neighbors to the East, West, North and South, respectively (if they exist); see Fig. 1. Then, the set $\mathbb{E}$ of edges of the graph is composed of the horizontal and vertical segments connecting the neighboring nodes, i.e., $(\mathbf{X}_i, \mathbf{X}_{i,\star})$ for $\star \in \{E,W,N,S\}$ and $i \in \mathbb{V}$. We shall denote $C_i^\star$ the conductivity of the edge $(\mathbf{X}_i, \mathbf{X}_{i,\star})$, and P_i, resp., $P_{i,\star}$, denote the pressure in the vertex $\mathbf{X}_i$, resp., $\mathbf{X}_{i,\star}$. Similarly, S_i^h denotes the source/sink in vertex $i \in \mathbb{V}$.

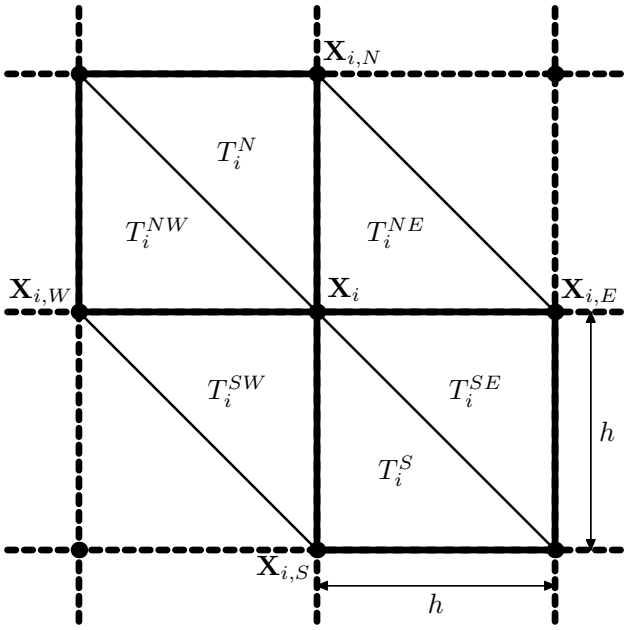

FIGURE 1. Interior node $\mathbf{X}_i$ with its four neighboring nodes $\mathbf{X}_{i,E}$, $\mathbf{X}_{i,W}$, $\mathbf{X}_{i,N}$, $\mathbf{X}_{i,S}$ and six adjacent triangles, T_i^{NE}, T_i^N, T_i^{NW}, T_i^{SW}, T_i^S, T_i^{SE}.

With this notation, the discrete energy functional (1.5) takes the particular form

$$E^h[C] = \frac{h^2}{2} \sum_{i \in \mathbb{V}} \sum_{\star \in \{E,W,N,S\}} (r + C_i^\star) \left(\frac{P_i - P_{i,\star}}{h} \right)^2 + \frac{\nu}{\gamma} (r + C_i^\star)^\gamma, \tag{4.1}$$

and the Kirchhoff law (1.6) is written as

$$\sum_{\star \in \{E,W,N,S\}} (r + C_i^\star) \frac{P_i - P_{i,\star}}{h} = h S_i^h. \qquad \text{for } i \in \mathbb{V}, \tag{4.2}$$

For reasons explained later, we shall restrict to the case $\gamma > 1$ in the sequel.

Our strategy is to perform a program analogous to the 1D case of Section 3: first, to put the discrete energy functionals (4.1) into an integral form and find an equivalence between solutions of the Kirchhoff law (4.2) and the above Poisson equation with appropriate conductivity. However, in the 2D case the situation is more complicated and we need to introduce a finite element discretization of the Poisson equation. We then establish a connection between the FE-discretization and the Kirchhoff law (4.2). In the next step we show the convergence of the sequence of reformulated discrete energy functionals towards a continuum one as $h = 1/N \to 0$, using standard results of the theory of finite elements. Finally, we introduce a diffusive term into the energy functional, which will allow us to construct global minimizers of the continuum energy functional.

4.1. **Finite element discretization of the Poisson equation.** We construct a regular triangulation on the domain Ω such that each interior node $\mathbf{X}_i$ has six adjacent triangles, T_i^{NE}, T_i^N, T_i^{NW}, T_i^{SW}, T_i^S, T_i^{SE}, see Fig. 1. Boundary nodes have three, two or only one adjacent triangles, depending on their location. The union of the triangles adjacent to each $\mathbf{X}_i$ is denoted by U_i. The collection of all triangles constructed in Ω is denoted by $\mathcal{T}^h$.

We fix $S \in L^2(\Omega)$ with $\int_\Omega S \,\mathrm{d}\mathbf{x} = 0$ and consider a discretization of the Poisson equation

$$-\nabla \cdot ((rI + c)\nabla p) = S \tag{4.3}$$

on Ω subject to the no-flux boundary conditions, using the first-order (piecewise linear) H^1 finite element method on the triangulation $\mathcal{T}^h$. Therefore, on each NE-triangle T_i^{NE} we construct the linear basis functions $\phi_{i;1}^{NE}$, $\phi_{i;2}^{NE}$, $\phi_{i;3}^{NE}$ with

$$\begin{aligned}
&\phi_{i;1}^{NE}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{NE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;1}^{NE}(\mathbf{X}_{i,N}) = 0,\\
&\phi_{i;2}^{NE}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{NE}(\mathbf{X}_{i,E}) = 1, \quad \phi_{i;2}^{NE}(\mathbf{X}_{i,N}) = 0,\\
&\phi_{i;3}^{NE}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{NE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;3}^{NE}(\mathbf{X}_{i,N}) = 1,
\end{aligned}$$

and analogously for the other triangles in U_i, see Section 5.1 of the Appendix for explicit formulae. Denoting $W^h \subset H^1(\Omega)$ the space of continuous, piecewise linear functions on the triangulation $\mathcal{T}^h$, the finite element discretization of (4.3) reads

$$\int_\Omega \nabla p^h \cdot (rI + c)\nabla \psi^h \,\mathrm{d}\mathbf{x} = \int_\Omega S\psi^h \,\mathrm{d}\mathbf{x} \qquad \text{for all } \psi^h \in W^h. \tag{4.4}$$

Using standard arguments (coercivity and continuity of the corresponding bilinear form) we construct a solution $p^h \in W^h$ of (4.4), unique up to an additive constant; without loss of generality we fix $\int_\Omega p^h(\mathbf{x}) \,\mathrm{d}\mathbf{x} = 0$. The solution is represented by its vertex values $P_i^h := p^h(\mathbf{X}_i)$, $i \in \mathbb{V}$. In particular, on each NE-triangle T_i^{NE} we have

$$p^h(\mathbf{x}) = P_i^h \phi_{i;1}^{NE}(\mathbf{x}) + P_{i,E}^h \phi_{i;2}^{NE}(\mathbf{x}) + P_{i,N}^h \phi_{i;3}^{NE}(\mathbf{x}), \qquad \mathbf{x} \in T_i^{NE},$$

and the gradient of p^h on T_i^{NE} is the constant vector

$$\nabla p^h(\mathbf{x}) = \frac{1}{h}(P_{i,E}^h - P_i^h, P_{i,N}^h - P_i^h), \qquad \mathbf{x} \in T_i^{NE}. \tag{4.5}$$

Analogous formulae hold for all other triangles in U_i, as explicitly listed in Section 5.2 of the Appendix.

We now establish a connection between the discretized Poisson equation (4.4) and the Kirchhoff law (4.2). For this purpose, we define the sequence of operators $\mathbb{Q}_0^h$ mapping the vector of conductivities $(C_i)_{i\in\mathbb{E}}$ onto piecewise constant 2×2 diagonal tensors,

$$\mathbb{Q}_0^h : (C_i)_{i\in\mathbb{E}} \mapsto \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \tag{4.6}$$

The functions $c_1 = c_1(\mathbf{x})$, $c_2 = c_2(\mathbf{x})$, defined on Ω, are constant on each triangle $T \in \mathcal{T}^h$ and c_1 takes the value of the conductivity of the horizontal edge of T and c_2 takes the value of the conductivity of the vertical edge of T. In particular, we have

$$c_1 := \begin{cases} C_i^E & \text{on } T_i^{NE}, \\ C_i^E & \text{on } T_i^{SE}, \\ C_i^W & \text{on } T_i^{SW}, \\ C_i^W & \text{on } T_i^{NW}, \end{cases} \qquad c_2 := \begin{cases} C_i^N & \text{on } T_i^{NE}, \\ C_i^S & \text{on } T_i^{S}, \\ C_i^S & \text{on } T_i^{SW}, \\ C_i^N & \text{on } T_i^{N}. \end{cases} \tag{4.7}$$

Then, for a given vector of conductivities $C = (C_i)_{i\in\mathbb{E}}$ we consider the discretized Poisson equation (4.4) with the conductivity tensor $c := \mathbb{Q}_0^h[C]$. For each $i \in \mathbb{V}$ we construct the test function ψ_i^h as

$$\psi_i^h := \phi_{i;1}^{NE} + \phi_{i;1}^{SE} + \phi_{i;1}^{S} + \phi_{i;1}^{SW} + \phi_{i;1}^{NW} + \phi_{i;1}^{N},$$

with the basis functions on the right-hand side defined in Section 5.1 of the Appendix. Consequently, each ψ_i^h is supported on U_i, linear on each triangle belonging to U_i, and continuous on Ω. Then, obviously, $\psi_i^h \in W^h$ and using it as a test function in (4.4), we calculate, for the triangle T_i^{NE},

$$\int_{T_i^{NE}} \nabla p^h \cdot \left(rI + \mathbb{Q}_0^h[C]\right) \nabla \psi_i^h \,\mathrm{d}\mathbf{x} = \frac{r + C_i^E}{2}\left(P_i^h - P_{i,E}^h\right) + \frac{r + C_i^N}{2}\left(P_i^h - P_{i,N}^h\right),$$

where we used (4.5), the identity $\nabla\psi_i^h \equiv -\frac{1}{h}(1,1)$ on T_i^{NE}, and orthogonality relations between gradients of the basis functions (for instance, $\nabla\phi_{i;2}^{NE}\cdot\nabla\phi_{i;3}^{NE} = 0$). Performing analogous calculations for the remaining triangles constituting U_i, namely, T_i^{SE}, T_i^S, T_i^{SW}, T_i^{NW} and T_i^N, see Section 5.3 of the Appendix for explicit details, we obtain

$$\int_\Omega \nabla p^h \cdot \left(rI + \mathbb{Q}_0^h[C]\right) \nabla \psi_i^h \,\mathrm{d}\mathbf{x} \;=\; \sum_{\star\in\{E,W,N,S\}} (r + C_i^\star)(P_i^h - P_{i,\star}^h). \tag{4.8}$$

Consequently, (4.4) gives the identity

$$\sum_{\star\in\{E,W,N,S\}} (r + C_i^\star)\frac{P_i^h - P_{i,\star}^h}{h} = \frac{1}{h}\int_\Omega S\psi_i^h \,\mathrm{d}\mathbf{x}$$

for all $i \in \mathbb{V}$. Thus, defining

$$S_i^h := \frac{1}{h^2}\int_\Omega S\psi_i^h \,\mathrm{d}\mathbf{x}, \tag{4.9}$$

we have the following result:

Lemma 4. *For any vector of nonnegative conductivities $C = (C_i)_{i\in\mathbb{E}}$ and $S \in L^2(\Omega)$ with $\int_\Omega S\,\mathrm{d}\mathbf{x} = 0$, let $p^h \in W^h$ be a solution of the finite element discretization* (4.4) *with $c := \mathbb{Q}_0^h[C]$. Then, $P_i^h := p^h(\mathbf{X}_i)$, $i \in \mathbb{V}$, is a solution of the rescaled Kirchhoff law* (4.2) *with the source/sink terms S_i^h given by* (4.9).

Note that since $\frac{1}{h^2}\int_\Omega \psi_i^h(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1$, and, by assumption, $S \in L^2(\Omega)$, the Lebesgue differentiation theorem gives

$$S_i^h = \frac{1}{h^2}\int_\Omega S\psi_i^h \,\mathrm{d}\mathbf{x} \to S(\overline{\mathbf{x}}) \qquad \text{for a.e. } \overline{\mathbf{x}} = \mathbf{X}_i \text{ as } h = 1/N \to 0.$$

Consequently, $(S_i^h)_{h>0}$ is an approximating sequence for the datum $S = S(\mathbf{x})$.

4.2. **Reformulation of the discrete energy functional.** We reformulate the energy functionals (4.1)–(4.2) such that they are defined on the space $L^\infty_+(\Omega)^{2\times2}_{\mathrm{diag}}$ of essentially bounded diagonal nonnegative tensors on Ω. We define the functional $\mathcal{E}^h : L^\infty_+(\Omega)^{2\times2}_{\mathrm{diag}} \to \mathbb{R}$,

$$\mathcal{E}^h[c] := \int_\Omega \nabla p^h[c]\cdot(rI + c)\nabla p^h[c] + \frac{\nu}{\gamma}\left(|r + c_1|^\gamma + |r + c_2|^\gamma\right)\,\mathrm{d}\mathbf{x}, \tag{4.10}$$

where $p^h[c] \in W^h$ is a solution of the finite element problem (4.4).

Proposition 3. *Let $S \in L^2(\Omega)$ with $\int_\Omega S\,\mathrm{d}\mathbf{x} = 0$ and S_i^h be given by* (4.9). *Then for any vector of nonnegative conductivities $C = (C_i)_{i\in E^N}$, we have*

$$E^h[C] = \mathcal{E}^h[\mathbb{Q}_0^h[C]],$$

with E^h defined in (4.1) *and $\mathcal{E}^h$ given by* (4.10).

Proof. We have shown in Section 4.1 that if $p^h = p^h(\mathbf{x})$ denotes a solution of the finite element problem (4.4) with $c = \mathbb{Q}_0^h[C]$, then the vertex values $P_i^h := p^h(\mathbf{X}_i)$ satisfy the Kirchhoff law (4.2). Moreover, using (4.5) and the definition (4.6)–(4.7) of $\mathbb{Q}_0^h[C]$, we calculate

$$\int_{T_i^{NE}} \nabla p^h \cdot (rI + \mathbb{Q}_0^h[C]) \nabla p^h \, \mathrm{d}\mathbf{x} = |T_i^{NE}| \left((r + C_i^E) \left(\frac{P_{i,E}^h - P_i^h}{h} \right)^2 + (r + C_i^N) \left(\frac{P_{i,N}^h - P_i^h}{h} \right)^2 \right)$$

for each $i \in \mathbb{V}$, and analogously for all other triangles. Noting that $|T_i^{NE}| = h^2/2$ and summing over all triangles, we obtain the formula (4.1) for the discrete energy $E^h[C]$.

■

4.3. **Convergence of the energy functional.** With Proposition 3, our task is now to prove the convergence of the sequence of functionals $\mathcal{E}^h$ given by (4.10) towards

$$\mathcal{E}[c] := \int_\Omega \nabla p[c] \cdot (rI + c) \nabla p[c] + \frac{\nu}{\gamma} \left(|r + c_1|^\gamma + |r + c_2|^\gamma \right) \mathrm{d}\mathbf{x}, \tag{4.11}$$

where $p[c] \in H^1(\Omega)$ is a weak solution of the Poisson equation (4.3) subject to no-flux boundary conditions, and c_1, c_2 are the diagonal entries of $c = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$. Similarly as in Section 3.2 we choose to work in the space $L_+^\infty(\Omega)_{\mathrm{diag}}^{2\times 2}$ of diagonal nonnegative tensors on Ω with essentially bounded entries, equipped with the norm topology of $L^2(\Omega)$. Note that for $c \in L_+^\infty(\Omega)_{\mathrm{diag}}^{2\times 2}$ the Poisson equation (4.3) has a solution $p[c] \in H^1(\Omega)$, unique up to an additive constant, and $\mathcal{E}[c] < +\infty$.

Lemma 5. *For any sequence of nonnegative diagonal tensors $(c^N)_{N\in\mathbb{N}} \subset L_+^\infty(\Omega)_{\mathrm{diag}}^{2\times 2}$ with entries uniformly bounded in $L^\gamma(\Omega)$ and converging entrywise to $c \in L_+^\gamma(\Omega)_{\mathrm{diag}}^{2\times 2}$ in the norm topology of $L^2(\Omega)$ as $h = 1/N \to 0$, we have,*

$$\mathcal{E}[c] \le \liminf_{h=1/N\to 0} \mathcal{E}^h[c^N], \tag{4.12}$$

with $\mathcal{E}^h$ given by (4.10) *and $\mathcal{E}$ defined in* (4.11).

Proof. Due to the strong convergence of the entries of c^N in $L^2(\Omega)$ there exist a subsequence converging almost everywhere in Ω to c. Then, we have by the Fatou Lemma,

$$\int_\Omega |r + c_1|^\gamma \, \mathrm{d}\mathbf{x} \le \liminf_{h=1/N\to 0} \int_\Omega |r + c_1^N|^\gamma \, \mathrm{d}\mathbf{x}, \tag{4.13}$$

which is finite due to the uniform boundedness of c_1^N in $L^\gamma(\Omega)$. Similarly for c_2^N.

For the sequel let us denote $p := p[c] \in H^1(\Omega)$ is a solution of the Poisson equation (4.3) with conductivity c, $p^N := p[c^N]$ a solution of the Poisson equation (4.3) with conductivity c^N and $p^h := p^h[c^N] \in W^h$ a solution of the finite element discretization (4.4) with $h = 1/N$ and conductivity c^N. Then, by an obvious modification of the auxiliary Lemma 1 for diagonal tensor-valued conductivities we have by (2.3),

$$\int_\Omega \nabla p \cdot (rI + c) \nabla p \, \mathrm{d}\mathbf{x} = \lim_{N\to\infty} \int_\Omega \nabla p^N \cdot (rI + c^N) \nabla p^N \, \mathrm{d}\mathbf{x}. \tag{4.14}$$

Let us define the bilinear forms $B^N : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$,

$$B^N(u, v) = \int_\Omega \nabla u \cdot (rI + c^N) \nabla v \, \mathrm{d}x.$$

Note that $B^N(u,v) < +\infty$ for $u,\ v \in H^1(\Omega)$ since $c^N \in L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$. Moreover, since $rI + c^N$ is symmetric and positive definite, B^N induces a seminorm on $H^1(\Omega)$,

$$|u|_{B^N} := \sqrt{B^N(u,u)} \qquad \text{for } u \in H^1(\Omega).$$

With this notation we have

$$\int_\Omega \nabla p^N \cdot (rI + c^N)\nabla p^N \,\mathrm{d}\mathbf{x} = \left|p^N\right|^2_{B^N}.$$

We now proceed along the lines of standard theory of the finite element method (proof of Céaś Lemma in the energy norm, see, e.g., [5]). Due to the Galerkin orthogonality

$$B^N(p^N - p^h, \psi) = 0 \qquad \text{for all } \psi \in W^h, \tag{4.15}$$

we have, noting that $p^h \in W^h$,

$$\left|p^N\right|^2_{B^N} = \left|p^N - p^h\right|^2_{B^N} + \left|p^h\right|^2_{B^N}.$$

Then, again by (4.15) and by the Cauchy-Schwartz inequality, we have for all $\psi \in W^h$,

$$\left|p^N - p^h\right|^2_{B^N} = B^N(p^N - p^h, p^N - \psi) \le \left|p^N - p^h\right|_{B^N} \left|p^N - \psi\right|_{B^N}.$$

Therefore, with the triangle inequality,

$$\left|p^N - p^h\right|_{B^N} \le \inf_{\psi\in W^h} \left|p^N - \psi\right|_{B^N} \le \left|p^N - p\right|_{B^N} + \inf_{\psi\in W^h} |p - \psi|_{B^N}.$$

Due to the strong convergence of $c^N \to c$ in $L^2(\Omega)$ and the standard result of approximation theory, see, e.g., [5], we have

$$\begin{aligned}
\lim_{h=1/N\to 0} \inf_{\psi\in W^h} |p-\psi|^2_{B^N} &\le \lim_{h\to 0} \inf_{\psi\in W^h} \int_\Omega \nabla(p-\psi)\cdot(rI+c)\nabla(p-\psi)\,\mathrm{d}\mathbf{x} \\
&\quad + \lim_{N\to\infty} \int_\Omega \nabla p\cdot(c^N - c)\nabla p\,\mathrm{d}\mathbf{x} \\
&= 0.
\end{aligned}$$

Due to (4.14) and the weak convergence of $p^N \rightharpoonup p$ in $H^1(\Omega)$,

$$\lim_{N\to\infty} \left|p^N - p\right|_{B^N} = 0. \tag{4.16}$$

Thus, collecting the above results from (4.14) up to (4.16), we conclude that

$$\int_\Omega \nabla p\cdot(rI+c)\nabla p\,\mathrm{d}\mathbf{x} = \lim_{N\to\infty} \left|p^N\right|^2_{B^N} = \lim_{h=1/N\to 0} \left|p^h\right|^2_{B^N} = \lim_{h=1/N\to 0} \int_\Omega \nabla p^h\cdot(rI+c^N)\nabla p^h\,\mathrm{d}\mathbf{x},$$

which together with (4.13) gives (4.12).

■

Remark 2. *Note that if $\gamma > 1$ and with the assumption that the sequence $(c^N)_{N\in\mathbb{N}}$ converges (entrywise) in the norm topology of $L^\gamma(\Omega)$, the statement of Lemma 5 can be strengthened to*

$$\mathcal{E}[c] = \lim_{h=1/N\to 0} \mathcal{E}^h[c^N].$$

This follows directly from the fact that in this case we have for the metabolic term

$$\int_\Omega |r+c_1|^\gamma + |r+c_2|^\gamma\,\mathrm{d}\mathbf{x} = \lim_{h=1/N\to 0} \int_\Omega |r+c_1^N|^\gamma + |r+c_2^N|^\gamma\,\mathrm{d}\mathbf{x}.$$

Lemma 5 and Remark 2 trivially imply the Γ-convergence of the sequence of energy functionals $\mathcal{E}^h$ in the norm topology of $L^\gamma(\Omega)$ for $\gamma > 1$:

Theorem 2. *Let $\gamma > 1$, $S \in L^2(\Omega)$ with $\int_\Omega S \,\mathrm{d}\mathbf{x} = 0$ and S_i^h be given by (4.9). Then the sequence $\mathcal{E}^h$ given by (4.10) Γ-converges to $\mathcal{E}$ defined in (4.11) with respect to the norm topology of $L^\gamma(\Omega)$ on the set $L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$. In particular:*

- *For any sequence $(c^N)_{N\in\mathbb{N}} \subset L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ converging entrywise to $c \in L^\gamma_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ in the norm topology of $L^\gamma(\Omega)$ as $h = 1/N \to 0$, we have*
$$\mathcal{E}[c] \le \liminf_{h=1/N\to 0} \mathcal{E}^h[c^N].$$
- *For any $c \in L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ there exists a sequence $(c^N)_{N\in\mathbb{N}} \subset L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ converging entrywise to $c \in L^\gamma_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ in the norm topology of $L^\gamma(\Omega)$ as $h = 1/N \to 0$, such that*
$$\mathcal{E}[c] \ge \limsup_{h=1/N\to 0} \mathcal{E}^h[c^N].$$

Proof. The lim inf-statement follows directly from Lemma 5. For the lim sup-statement it is sufficient to set $c^N := c$ for all $N \in \mathbb{N}$ and use Remark 2, which in fact leads to the stronger statement
$$\mathcal{E}[c] = \lim_{h=1/N\to 0} \mathcal{E}^h[c^N].$$
∎

4.4. **Introduction of diffusion and construction of continuum energy minimizers ($\gamma > 1$).** As in the one-dimensional case, we introduce a diffusive term into the discrete energy functionals, which shall provide compactness of the sequence of energy minimizers. We again construct a piecewise linear approximation of the discrete conductivities C, which, however, turns out to be technically quite involved in the two-dimensional situation.

We shall describe the process for the conductivities of the horizontal edges, and by a slight abuse of notation, we denote $C_{i+1/2,j}$ the conductivity of the horizontal edge connecting the node (ih, jh) to $((i+1)h, jh)$ for $i = 0, \dots, N-1, j = 0, \dots, N$ where $h = 1/N$. Moreover, we denote $\mathbf{M}_{i+1/2,j}$ the midpoint of this edge, i.e., $\mathbf{M}_{i+1/2,j} = ((i+1/2)h, jh)$. For a given vector of conductivities C, we construct the continuous function $\mathbb{Q}_1^h[C]$ on Ω, such that

$$\mathbb{Q}_1^h[C](\mathbf{M}_{i+1/2,,j}) = C_{i+1/2,j}, \qquad \text{for } i = 0, \dots, N-1,\ j = 0, \dots, N,$$

and $\mathbb{Q}_1^h[C]$ is linear on each triangle spanned by the nodes $\mathbf{M}_{i-1/2,j}$, $\mathbf{M}_{i+1/2,j}$, $\mathbf{M}_{i-1/2,j+1}$ and on each triangle spanned by the nodes $\mathbf{M}_{i+1/2,j}$, $\mathbf{M}_{i+1/2,j+1}$, $\mathbf{M}_{i-1/2,j+1}$, for $i = 1, \dots, N-1,\ j = 0, \dots, N-1$. Let us denote the union of such two triangles, i.e., the square spanned by the nodes $\mathbf{M}_{i-1/2,j}$, $\mathbf{M}_{i+1/2,j}$, $\mathbf{M}_{i-1/2,j+1}$ and $\mathbf{M}_{i+1/2,j+1}$, by W_{ij}. Then, a simple calculation reveals that

$$\int_{W_{ij}} |\nabla \mathbb{Q}_1^h[C]|^2 \,\mathrm{d}\mathbf{x} = \frac{1}{2}\left[(C_{i-1/2,j} - C_{i+1/2,j})^2 + (C_{i+1/2,j} - C_{i+1/2,j+1})^2 + (C_{i+1/2,j+1} - C_{i-1/2,j+1})^2 + (C_{i-1/2,j+1} - C_{i-1/2,j})^2\right]. \tag{4.17}$$

On the "boundary stripe" $(0, h/2) \times (0,1)$ and $(1-h/2, 1) \times (0,1)$ the function is defined to be constant in the x-direction, such that it is globally continuous on Ω, i.e.,

$$\mathbb{Q}_1^h[C](\mathbf{x}) := \frac{C_{1/2,j+1} - C_{1/2,j}}{h}(x_2 - jh) + C_{1/2,j}, \qquad \text{for } \mathbf{x} = (x_1, x_2) \in (0, h/2) \times (jh, (j+1)h),$$

$$\mathbb{Q}_1^h[C](\mathbf{x}) := \frac{C_{N-1/2,j+1} - C_{N-1/2,j}}{h}(x_2 - jh) + C_{N-1/2,j}, \qquad \text{for } \mathbf{x} = (x_1, x_2) \in (1-h/2, 1) \times (jh, (j+1)h)$$

for $j = 0, \dots, N-1$. Summing up (4.17) over all squares W_{ij} and the boundary stripe, we arrive at

$$\int_\Omega |\nabla \mathbb{Q}_1^h[C]|^2 \,\mathrm{d}\mathbf{x} = \mathbb{D}_x[C], \tag{4.18}$$

with

$$\begin{aligned}\mathbb{D}_x[C] := &\sum_{i=0}^{N-1}\sum_{j=0}^{N-1}(C_{i+1/2,j} - C_{i+1/2,j+1})^2 + \sum_{i=1}^{N-1}\sum_{j=1}^{N-1}(C_{i-1/2,j} - C_{i+1/2,j})^2 \\ &+\frac{1}{2}\sum_{i=1}^{N-1}\left[(C_{i-1/2,0} - C_{i+1/2,0})^2 + (C_{i-1/2,N} - C_{i+1/2,N})^2\right].\end{aligned}$$

Performing the same procedure for the vertical edges, we obtain

$$\int_\Omega |\nabla \mathbb{Q}_2^h[C]|^2 \,\mathrm{d}\mathbf{x} = \mathbb{D}_y[C], \tag{4.19}$$

with obvious definitions of $\mathbb{Q}_2^h[C]$ and $\mathbb{D}_y[C]$.

Consequently, we define the sequence of discrete energy functionals E^h_{diff},

$$E^h_{\mathrm{diff}}[C] := D^2\left(\mathbb{D}_x[C] + \mathbb{D}_y[C]\right) + E^h[C], \tag{4.20}$$

with $D^2 > 0$ diffusion constant and $E^h[C]$ defined in (4.1), coupled to the Kirchhoff law (4.2) with sources/sinks S_i^h given by (4.9). We then have:

Proposition 4. *For any vector $C = (C_i)_{i\in\mathbb{E}}$ of nonnegative entries, we have*

$$E^h_{\mathrm{diff}}[C] = D^2\int_\Omega |\nabla \mathbb{Q}_1^h[C]|^2 + |\nabla \mathbb{Q}_2^h[C]|^2 \,\mathrm{d}\mathbf{x} + \mathcal{E}^h[\mathbb{Q}_0^h[C]],$$

with E^h_{diff} defined in (4.20) *and $\mathcal{E}^h$ given by* (4.10) *with the pressures p^h being a solution of the FEM-discretized Poisson equation* (4.4) *with $c = \mathbb{Q}_0^N[C]$.*

We are now in shape to prove the main result of this section:

Theorem 3. *Let $\gamma > 1$, $S \in L^2(\Omega)$ with $\int_\Omega S \,\mathrm{d}\mathbf{x} = 0$ and S_i^h be given by* (4.9). *Let $(C^N)_{N\in\mathbb{N}} \subset \mathbb{R}^N$ be a sequence of global minimizers of the discrete energy functionals E^h_{diff} given by* (4.20) *with $h = 1/N$. Then the sequence of diagonal 2×2 matrices*

$$c^N := \begin{pmatrix}\mathbb{Q}_1^h[C^N] & 0\\ 0 & \mathbb{Q}_2^h[C^N]\end{pmatrix}$$

converges weakly in $H^1(\Omega)^{2\times 2}$ to $c \in H^1(\Omega)^{2\times 2}_+$ as $h = 1/N \to 0$, with c a global minimizer of the functional $\mathcal{E}_{\mathrm{diff}} : H^1_+(\Omega)^{2\times 2}_{\mathrm{diag}} \to \mathbb{R}$,

$$\mathcal{E}_{\mathrm{diff}}[c] := D^2\int_\Omega |\nabla c_1|^2 + |\nabla c_2|^2 \,\mathrm{d}\mathbf{x} + \mathcal{E}[c],$$

where $\mathcal{E}[c]$ is given by (4.11).

Proof. Let us observe that

$$E^h_{\mathrm{diff}}[C^N] \le E^h_{\mathrm{diff}}[0] = \frac{h^2}{2}\sum_{i\in\mathbb{V}}\sum_{\star\in\{E,W,N,S\}} r\left(\frac{\widetilde{P}_i - \widetilde{P}_{i,\star}}{h}\right)^2 + \frac{\nu}{\gamma}r^\gamma,$$

where $(\widetilde{P}_i)_{i\in\mathbb{V}}$ is a solution of the Kirchhoff law (4.2) with conductivities $C = 0$ and sources/sinks given by (4.9). As shown in Section 4.1, the pressures $\widetilde{P}_i$ correspond to pointwise values $\widetilde{P}_i^h :=$

$\tilde{p}^h(\mathbf{X}_i)$, $i \in \mathbb{V}$, of the solution $\tilde{p}^h$ of the discretized Poisson equation (4.4) with conductivity tensor $c = 0$. Moreover, due to formula (4.5) we have

$$\frac{h^2}{2} \sum_{i \in \mathbb{V}} \sum_{\star \in \{E,W,N,S\}} r \left(\frac{\widetilde{P}_i - \widetilde{P}_{i,\star}}{h} \right)^2 = r \int_\Omega |\nabla \tilde{p}^h|^2 \, \mathrm{d}\mathbf{x},$$

and the uniform boundedness of $\nabla \tilde{p}^h$ in $L^2(\Omega)$ implies a uniform bound on $E^h_{\mathrm{diff}}[C^N]$.

Since the sequence

$$D^2 \int_\Omega |\nabla \mathbb{Q}^h_1[C^N]|^2 + |\nabla \mathbb{Q}^h_2[C^N]|^2 \, \mathrm{d}\mathbf{x} = D^2 \left(\mathbb{D}_x[C^N] + \mathbb{D}_y[C^N] \right) \leq E^h_{\mathrm{diff}}[C^N]$$

is uniformly bounded, there exist subsequences of $\mathbb{Q}^h_1[C^N]$ and $\mathbb{Q}^h_2[C^N]$ converging to some c_1, $c_2 \in H^1(\Omega)$ weakly in $H^1(\Omega)$, and strongly in $L^2(\Omega)$. It is easy to check that then also $\mathbb{Q}^h_0[C^N]$ converges to $c := \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}$ strongly in $L^2(0,1)^{2\times 2}$. Clearly, we also have $\mathbb{Q}^h_0[C^N] \in L^\infty_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ with entries uniformly bounded in $L^\gamma(\Omega)$. Consequently, by Lemma 5, we have

$$\mathcal{E}[c] \leq \liminf_{h=1/N \to 0} \mathcal{E}^h[c^N].$$

Moreover, due to the weak lower semicontinuity of the L^2-norm, we have

$$\int_\Omega |\nabla c_1|^2 + |\nabla c_2|^2 \, \mathrm{d}\mathbf{x} \leq \liminf_{h=1/N \to 0} \int_\Omega |\nabla \mathbb{Q}^h_1[C^N]|^2 + |\nabla \mathbb{Q}^h_2[C^N]|^2 \, \mathrm{d}\mathbf{x}\,.$$

Consequently,

$$\mathcal{E}_{\mathrm{diff}}[c] \leq \liminf_{h=1/N \to 0} E^h_{\mathrm{diff}}[C^N]. \tag{4.21}$$

We claim that c is a global minimizer of $\mathcal{E}_{\mathrm{diff}}$ in $H^1_+(\Omega)^{2\times 2}_{\mathrm{diag}}$. For contradiction, assume that there exists $\overline{c} \in H^1_+(\Omega)^{2\times 2}_{\mathrm{diag}}$ such that

$$\mathcal{E}_{\mathrm{diff}}[\overline{c}] < \mathcal{E}_{\mathrm{diff}}[c].$$

We define the sequence $(\overline{C}^N)_{N\in\mathbb{N}}$ by setting the conductivity $\overline{C}^N_i$ of each horizontal edge $i \in \mathbb{E}$ to the average of $\overline{c}_1$ over the two triangles $T_{i;1}$, $T_{i;2} \in \mathcal{T}^h$ that contain the edge i, i.e.,

$$\overline{C}^N_i := \frac{1}{h^2} \int_{T_{i;1} \cup T_{i;2}} \overline{c}_1(x) \, \mathrm{d}\mathbf{x}.$$

Similarly, we use the averages of $\overline{c}_2$ to define the conductivities of the vertical edges. Then, by assumption, we have for all $h = 1/N$, $N \in \mathbb{N}$,

$$E^h_{\mathrm{diff}}[\overline{C}^N] \geq E^h_{\mathrm{diff}}[C^N]. \tag{4.22}$$

It is easy to check that the sequence $\mathbb{Q}^h_1[\overline{C}^N]$ converges strongly in $H^1(\Omega)$ towards $\overline{c}_1$, therefore

$$\int_\Omega |\nabla \mathbb{Q}^h_1[\overline{C}^N]|^2 \, \mathrm{d}\mathbf{x} \to \int_\Omega |\nabla \overline{c}_1|^2 \, \mathrm{d}\mathbf{x} \qquad \text{as } h = 1/N \to 0,$$

and analogously for $\mathbb{Q}^h_2[\overline{C}^N]$ and $\overline{c}_2$. Moreover, the sequence $\mathbb{Q}^h_0[\overline{C}^N]$ converges to $\overline{c}$ strongly in $L^\gamma(\Omega)^{2\times 2}_{\mathrm{diag}}$, therefore, by Remark 2, $\mathcal{E}^h[\mathbb{Q}^N_0[\overline{C}^N]] \to \mathcal{E}[\overline{c}]$ as $h = 1/N \to 0$. Consequently,

$$\lim_{h=1/N \to 0} E^h_{\mathrm{diff}}[\overline{C}^N] = \mathcal{E}_{\mathrm{diff}}[\overline{c}] < \mathcal{E}_{\mathrm{diff}}[c],$$

a contradiction to (4.21)–(4.22).

■

Remark 3. *We can easily generalize to the situation when the two-dimensional grid is not rectangular, but consists of parallelograms with sides of equal length in linearly independent directions* θ_1, $\theta_2 \in \mathbb{S}^1$, *where* $\mathbb{S}^1$ *is the unit circle in* $\mathbb{R}^2$. *Then the coordinate transform*

$$(1,0) \mapsto \theta_1, \qquad (0,1) \mapsto \theta_2$$

in (4.11) *leads to the transformed continuum energy functional*

$$\mathcal{E}[c] = \int_\Omega \nabla p[c] \cdot \mathbb{P}[c] \nabla p[c] + \frac{\nu}{\gamma} \left(|r + c_1|^\gamma + |r + c_2|^\gamma \right) \mathrm{d}\mathbf{x}$$

coupled to the Poisson equation

$$-\nabla \cdot (\mathbb{P}[c] \nabla p) = S$$

with the permeability tensor

$$\mathbb{P}[c] = rI + c_1 \theta_1 \otimes \theta_1 + c_2 \theta_2 \otimes \theta_2.$$

The eigenvalues of $\mathbb{P}[c]$ *(principal permeabilities) are*

$$\lambda_{1,2} = \frac{1}{2} \left(c_1 + c_2 \pm \sqrt{(c_1 - c_2)^2 - 4 c_1 c_2 (\theta_1 \cdot \theta_2)^2} \right)$$

and the corresponding eigenvectors (principal directions)

$$u_{1,2} = \theta_1 + \frac{c_2 - c_1 \pm \sqrt{(c_1 - c_2)^2 - 4 c_1 c_2 (\theta_1 \cdot \theta_2)^2}}{2 c_1 \theta_1 \cdot \theta_2} \theta_2.$$

5. Appendix

Here we provide more technical details for the constructions and calculations performed in Section 4.1.

5.1. **Linear basis functions.** We list the explicit definitions for the piecewise linear basis functions on the triangulation $\mathcal{T}^h$, constructed in Section 4.1. Any interior node $i \in \mathbb{V}$ has six adjacent triangles, denoted clockwise by T_i^{NE}, T_i^{SE}, T_i^S, T_i^{SW}, T_i^{NW}, T_i^N, see Fig. 1. For each triangle we construct three basis functions, supported on the respective triangle and linear on their support. Obviously, the basis functions are uniquely determined by their values on the triangle vertices. For later reference we list their gradients, which are constant vectors on the respective triangles.

- On the NE-triangle T_i^{NE} we construct the linear basis functions $\phi_{i;1}^{NE}$, $\phi_{i;2}^{NE}$, $\phi_{i;3}^{NE}$ defined by

$$\begin{aligned} &\phi_{i;1}^{NE}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{NE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;1}^{NE}(\mathbf{X}_{i,N}) = 0, \\ &\phi_{i;2}^{NE}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{NE}(\mathbf{X}_{i,E}) = 1, \quad \phi_{i;2}^{NE}(\mathbf{X}_{i,N}) = 0, \\ &\phi_{i;3}^{NE}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{NE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;3}^{NE}(\mathbf{X}_{i,N}) = 1, \end{aligned}$$

  so that

$$\nabla \phi_{i;1}^{NE} \equiv -\frac{1}{h}(1,1), \qquad \nabla \phi_{i;2}^{NE} \equiv \frac{1}{h}(1,0), \qquad \nabla \phi_{i;3}^{NE} \equiv \frac{1}{h}(0,1), \qquad \text{on } T_i^{NE}.$$

- On the SE-triangle T_i^{SE} we construct the linear basis functions $\phi_{i;1}^{SE}$, $\phi_{i;2}^{SE}$, $\phi_{i;3}^{SE}$ defined by

$$\begin{aligned}
&\phi_{i;1}^{SE}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{SE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;1}^{SE}(\mathbf{X}_{i,SE}) = 0,\\
&\phi_{i;2}^{SE}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{SE}(\mathbf{X}_{i,E}) = 1, \quad \phi_{i;2}^{SE}(\mathbf{X}_{i,SE}) = 0,\\
&\phi_{i;3}^{SE}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{SE}(\mathbf{X}_{i,E}) = 0, \quad \phi_{i;3}^{SE}(\mathbf{X}_{i,SE}) = 1,
\end{aligned}$$

  so that

$$\nabla\phi_{i;1}^{SE} \equiv -\frac{1}{h}(1,0), \qquad \nabla\phi_{i;2}^{SE} \equiv \frac{1}{h}(1,1), \qquad \nabla\phi_{i;3}^{SE} \equiv -\frac{1}{h}(0,1), \qquad \text{on } T_i^{SE}.$$

- On the S-triangle T_i^{S} we construct the linear basis functions $\phi_{i;1}^{S}$, $\phi_{i;2}^{S}$, $\phi_{i;3}^{S}$ defined by

$$\begin{aligned}
&\phi_{i;1}^{S}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{S}(\mathbf{X}_{i,SE}) = 0, \quad \phi_{i;1}^{S}(\mathbf{X}_{i,S}) = 0,\\
&\phi_{i;2}^{S}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{S}(\mathbf{X}_{i,SE}) = 1, \quad \phi_{i;2}^{S}(\mathbf{X}_{i,S}) = 0,\\
&\phi_{i;3}^{S}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{S}(\mathbf{X}_{i,SE}) = 0, \quad \phi_{i;3}^{S}(\mathbf{X}_{i,S}) = 1,
\end{aligned}$$

  so that

$$\nabla\phi_{i;1}^{S} \equiv \frac{1}{h}(0,1), \qquad \nabla\phi_{i;2}^{S} \equiv \frac{1}{h}(1,0), \qquad \nabla\phi_{i;3}^{S} \equiv -\frac{1}{h}(1,1), \qquad \text{on } T_i^{S}.$$

- On the SW-triangle T_i^{SW} we construct the linear basis functions $\phi_{i;1}^{SW}$, $\phi_{i;2}^{SW}$, $\phi_{i;3}^{SW}$ defined by

$$\begin{aligned}
&\phi_{i;1}^{SW}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{SW}(\mathbf{X}_{i,S}) = 0, \quad \phi_{i;1}^{SW}(\mathbf{X}_{i,W}) = 0,\\
&\phi_{i;2}^{SW}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{SW}(\mathbf{X}_{i,S}) = 1, \quad \phi_{i;2}^{SW}(\mathbf{X}_{i,W}) = 0,\\
&\phi_{i;3}^{SW}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{SW}(\mathbf{X}_{i,S}) = 0, \quad \phi_{i;3}^{SW}(\mathbf{X}_{i,W}) = 1,
\end{aligned}$$

  so that

$$\nabla\phi_{i;1}^{SW} \equiv \frac{1}{h}(1,1), \qquad \nabla\phi_{i;2}^{SW} \equiv -\frac{1}{h}(0,1), \qquad \nabla\phi_{i;3}^{SW} \equiv -\frac{1}{h}(1,0), \qquad \text{on } T_i^{SW}.$$

- On the NW-triangle T_i^{NW} we construct the linear basis functions $\phi_{i;1}^{NW}$, $\phi_{i;2}^{NW}$, $\phi_{i;3}^{NW}$ defined by

$$\begin{aligned}
&\phi_{i;1}^{NW}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{NW}(\mathbf{X}_{i,W}) = 0, \quad \phi_{i;1}^{NW}(\mathbf{X}_{i,NW}) = 0,\\
&\phi_{i;2}^{NW}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{NW}(\mathbf{X}_{i,W}) = 1, \quad \phi_{i;2}^{NW}(\mathbf{X}_{i,NW}) = 0,\\
&\phi_{i;3}^{NW}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{NW}(\mathbf{X}_{i,W}) = 0, \quad \phi_{i;3}^{NW}(\mathbf{X}_{i,NW}) = 1,
\end{aligned}$$

  so that

$$\nabla\phi_{i;1}^{NW} \equiv \frac{1}{h}(1,0), \qquad \nabla\phi_{i;2}^{NW} \equiv -\frac{1}{h}(1,1), \qquad \nabla\phi_{i;3}^{NW} \equiv \frac{1}{h}(0,1), \qquad \text{on } T_i^{NW}.$$

- On the N-triangle T_i^{N} we construct the linear basis functions $\phi_{i;1}^{N}$, $\phi_{i;2}^{N}$, $\phi_{i;3}^{N}$ defined by

$$\begin{aligned}
&\phi_{i;1}^{N}(\mathbf{X}_i) = 1, \quad \phi_{i;1}^{N}(\mathbf{X}_{i,NW}) = 0, \quad \phi_{i;1}^{N}(\mathbf{X}_{i,N}) = 0,\\
&\phi_{i;2}^{N}(\mathbf{X}_i) = 0, \quad \phi_{i;2}^{N}(\mathbf{X}_{i,NW}) = 1, \quad \phi_{i;2}^{N}(\mathbf{X}_{i,N}) = 0,\\
&\phi_{i;3}^{N}(\mathbf{X}_i) = 0, \quad \phi_{i;3}^{N}(\mathbf{X}_{i,NW}) = 0, \quad \phi_{i;3}^{N}(\mathbf{X}_{i,N}) = 1,
\end{aligned}$$

  so that

$$\nabla\phi_{i;1}^{N} \equiv -\frac{1}{h}(0,1), \qquad \nabla\phi_{i;2}^{N} \equiv -\frac{1}{h}(1,0), \qquad \nabla\phi_{i;3}^{N} \equiv \frac{1}{h}(1,1), \qquad \text{on } T_i^{N}.$$

5.2. **Gradients of p^h.** Here we provide the gradient of the solution $p^h \in W^h$ of (4.4), constructed in Section 4.1. Since p^h is continuous on Ω and linear on each triangle in $\mathcal{T}^h$, it is represented by its vertex values $P_i^h := p^h(\mathbf{X}_i)$, $i \in \mathbb{V}$. Then, for any interior node $i \in \mathbb{V}$ we readily have

$$\nabla p^h = \frac{1}{h} \begin{cases} (P_{i,E}^h - P_i^h, P_{i,N}^h - P_i^h) & \text{on } T_i^{NE}, \\ (P_{i,E}^h - P_i^h, P_i^h - P_{i,SE}^h) & \text{on } T_i^{SE}, \\ (P_{i,SE}^h - P_i^h, P_i^h - P_{i,S}^h) & \text{on } T_i^{S}, \\ (P_i^h - P_{i,W}^h, P_i^h - P_{i,S}^h) & \text{on } T_i^{SW}, \\ (P_i^h - P_{i,W}^h, P_{i,NW}^h - P_i^h) & \text{on } T_i^{NW}, \\ (P_i^h - P_{i,NW}^h, P_{i,N}^h - P_i^h) & \text{on } T_i^{N}. \end{cases}$$

5.3. **Explicit calculation for** (4.8)**.** Finally, we provide the detailed calculation for the identity (4.8). Noting that ψ_i^h is supported on $U_i = T_i^{NE} \cup T_i^{SE} \cup T_i^S \cup T_i^{SW} \cup T_i^{NW} \cup T_i^N$, and taking into account the results listed in Sections 5.1 and 5.2, we have

$$\begin{aligned}
\int_{T_i^{NE}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^E}{2} \Big(P_i^h - P_{i,E}^h\Big) + \frac{r + C_i^N}{2} \Big(P_i^h - P_{i,N}^h\Big), \\
\int_{T_i^{SE}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^E}{2} \Big(P_i^h - P_{i,E}^h\Big), \\
\int_{T_i^{S}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^S}{2} \Big(P_i^h - P_{i,S}^h\Big), \\
\int_{T_i^{SW}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^W}{2} \Big(P_i^h - P_{i,W}^h\Big) + \frac{r + C_i^S}{2} \Big(P_i^h - P_{i,S}^h\Big), \\
\int_{T_i^{NW}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^W}{2} \Big(P_i^h - P_{i,W}^h\Big), \\
\int_{T_i^{N}} \nabla p^h \cdot \Big(rI + \mathbb{Q}_0^h[C]\Big) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= \frac{r + C_i^N}{2} \Big(P_i^h - P_{i,N}^h\Big).
\end{aligned}$$

Summing up, we arrive at

$$\begin{aligned}
\int_{\Omega} \nabla p^h \cdot (rI + \mathbb{Q}_0^h[C]) \nabla \psi_i^h \, \mathrm{d}\mathbf{x} &= (r + C_i^E) \Big(P_i^h - P_{i,E}^h\Big) + (r + C_i^N) \Big(P_i^h - P_{i,N}^h\Big) \\
&+ (r + C_i^W) \Big(P_i^h - P_{i,W}^h\Big) + (r + C_i^S) \Big(P_i^h - P_{i,S}^h\Big),
\end{aligned}$$

which is (4.8).

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