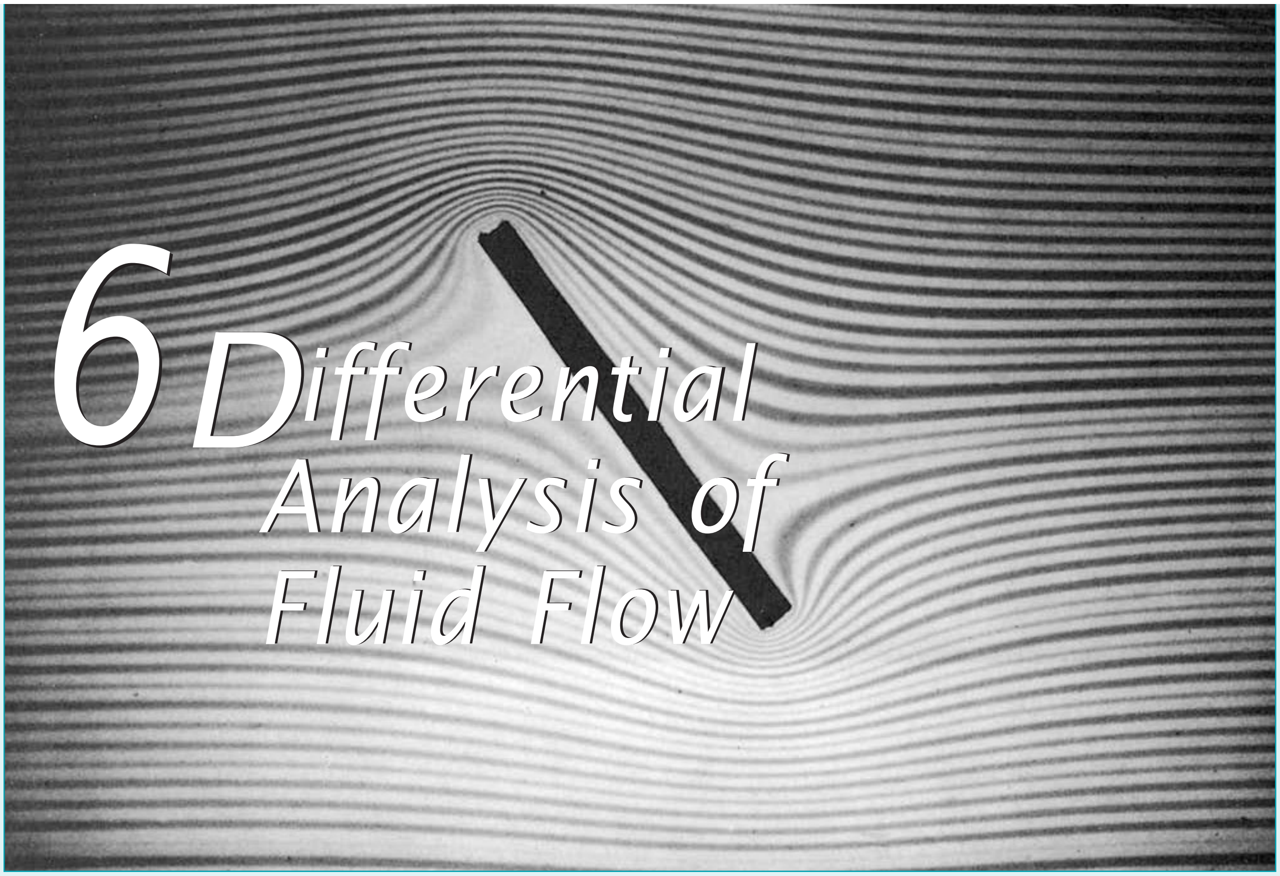
6 Differential Analysis of Fluid Flow



CHAPTER OPENING PHOTO: Flow past an inclined plate: The streamlines of a viscous fluid flowing slowly past a two-dimensional object placed between two closely spaced plates (a Hele-Shaw cell) approximate inviscid, irrotational (potential) flow. (Dye in water between glass plates spaced 1 mm apart.) (Photography courtesy of D. H. Peregrine.)

Learning Objectives

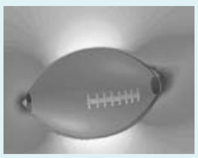
After completing this chapter, you should be able to:

- determine various kinematic elements of the flow given the velocity field.
- explain the conditions necessary for a velocity field to satisfy the continuity equation.
- apply the concepts of stream function and velocity potential.
- characterize simple potential flow fields.
- analyze certain types of flows using the Navier–Stokes equations.

In the previous chapter attention is focused on the use of finite control volumes for the solution of a variety of fluid mechanics problems. This approach is very practical and useful, since it does not generally require a detailed knowledge of the pressure and velocity variations within the control volume. Typically, we found that only conditions on the surface of the control volume were needed, and thus problems could be solved without a detailed knowledge of the flow field. Unfortunately, there are many situations that arise in which the details of the flow are important and the finite control volume approach will not yield the desired information. For example, we may need to know how the velocity varies over the cross section of a pipe, or how the pressure and shear stress vary along the surface of an airplane wing. In these circumstances we need to develop relationships that apply at a point, or at least in a very small infinitesimal region within a given flow field. This approach, which involves an *infinitesimal control volume*, as distinguished from a finite control volume, is commonly referred to as *differential analysis*, since (as we will soon discover) the governing equations are differential equations.

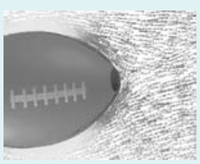
In this chapter we will provide an introduction to the differential equations that describe (in detail) the motion of fluids. Unfortunately, we will also find that these equations are rather complicated, non-linear partial differential equations that cannot be solved exactly except in a few cases, where simplifying assumptions are made. Thus, although differential analysis has the potential for supplying very detailed information about flow fields, this information is not easily extracted. Nevertheless, this approach provides a fundamental basis for the study of fluid mechanics. We do not want to be too discouraging at this point, since there are some exact solutions for laminar flow that can be obtained, and these have proved to be very useful. A few of these are included in this chapter. In addition, by making some simplifying assumptions many other analytical solutions can be obtained. For example, in some circumstances it may be reasonable to assume that the effect of viscosity is small and can be neglected. This rather drastic assumption greatly simplifies the analysis and provides the opportunity to obtain detailed solutions to a variety of complex flow problems. Some examples of these so-called *inviscid flow* solutions are also described in this chapter.

V6.1 Spinning football-velocity contours



It is known that for certain types of flows the flow field can be conceptually divided into two regions—a very thin region near the boundaries of the system in which viscous effects are important, and a region away from the boundaries in which the flow is essentially inviscid. By making certain assumptions about the behavior of the fluid in the thin layer near the boundaries, and using the assumption of inviscid flow outside this layer, a large class of problems can be solved using differential analysis. These boundary layer problems are discussed in Chapter 9. Finally, it is to be noted that with the availability of powerful computers it is feasible to attempt to solve the differential equations using the techniques of numerical analysis. Although it is beyond the scope of this book to delve extensively into this approach, which is generally referred to as *computational fluid dynamics* (CFD), the reader should be aware of this approach to complex flow problems. CFD has become a common engineering tool and a brief introduction can be found in Appendix A. To introduce the power of CFD, two animations based on the numerical computations are provided as shown in the margin.

V6.2 Spinning football-velocity vectors

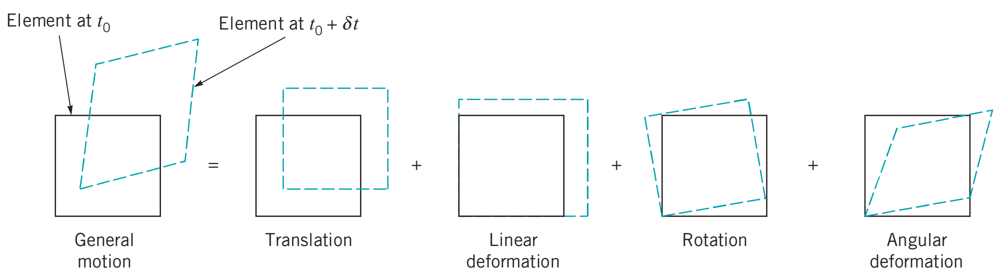


We begin our introduction to differential analysis by reviewing and extending some of the ideas associated with fluid kinematics that were introduced in Chapter 4. With this background the remainder of the chapter will be devoted to the derivation of the basic differential equations (which will be based on the principle of conservation of mass and Newton's second law of motion) and to some applications.

6.1 Fluid Element Kinematics

Fluid element motion consists of translation, linear deformation, rotation, and angular deformation.

In this section we will be concerned with the mathematical description of the motion of fluid elements moving in a flow field. A small fluid element in the shape of a cube which is initially in one position will move to another position during a short time interval δt as illustrated in Fig. 6.1. Because of the generally complex velocity variation within the field, we expect the element not only to translate from one position but also to have its volume changed (linear deformation), to rotate, and to undergo a change in shape (angular deformation). Although these movements and deformations occur simultaneously, we can consider each one separately as illustrated in Fig. 6.1. Since element motion and deformation are intimately related to the velocity and variation of velocity throughout the flow field, we will briefly review the manner in which velocity and acceleration fields can be described.

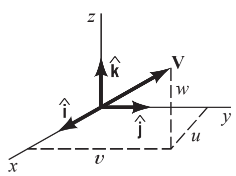


■ **FIGURE 6.1** **Types of motion and deformation for a fluid element.**

6.1.1 Velocity and Acceleration Fields Revisited

As discussed in detail in Section 4.1, the velocity field can be described by specifying the velocity **V** at all points, and at all times, within the flow field of interest. Thus, in terms of rectangular coordinates, the notation **V** (x, y, z, t) means that the velocity of a fluid particle depends on where it is located within the flow field (as determined by its coordinates, x, y, and z) and when it occupies the particular point (as determined by the time, t). As is pointed out in Section 4.1.1, this method of describing the fluid motion is called the Eulerian method. It is also convenient to express the velocity in terms of three rectangular components so that

$$\mathbf{V} = u\hat{\mathbf{i}} + v\hat{\mathbf{j}} + w\hat{\mathbf{k}} \tag{6.1}$$



where u, v, and w are the velocity components in the x, y, and z directions, respectively, and $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are the corresponding unit vectors, as shown by the figure in the margin. Of course, each of these components will, in general, be a function of x, y, z, and t. One of the goals of differential analysis is to determine how these velocity components specifically depend on x, y, z, and t for a particular problem.

With this description of the velocity field it was also shown in Section 4.2.1 that the acceleration of a fluid particle can be expressed as

$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \tag{6.2}$$

and in component form:

$$a_x = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{6.3a}$$

$$a_y = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{6.3b}$$

$$a_z = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{6.3c}$$

The acceleration of a fluid particle is described using the concept of the material derivative.

The acceleration is also concisely expressed as

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt} \tag{6.4}$$

where the operator

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z} \tag{6.5}$$

is termed the *material derivative*, or *substantial derivative*. In vector notation

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})(\ ) \tag{6.6}$$

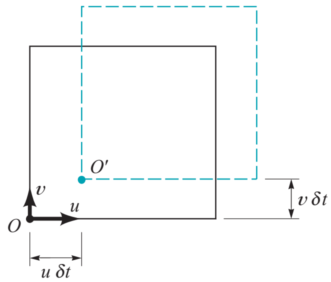
where the gradient operator, $\boldsymbol{\nabla}(\ )$, is

$$\boldsymbol{\nabla}(\ ) = \frac{\partial(\ )}{\partial x}\hat{\mathbf{i}} + \frac{\partial(\ )}{\partial y}\hat{\mathbf{j}} + \frac{\partial(\ )}{\partial z}\hat{\mathbf{k}} \tag{6.7}$$

which was introduced in Chapter 2. As we will see in the following sections, the motion and deformation of a fluid element depend on the velocity field. The relationship between the motion and the forces causing the motion depends on the acceleration field.

6.1.2 Linear Motion and Deformation

The simplest type of motion that a fluid element can undergo is translation, as illustrated in Fig. 6.2. In a small time interval δt a particle located at point O will move to point O' as is illustrated in the figure. If all points in the element have the same velocity (which is only true if there are no velocity gradients), then the element will simply translate from one position to another. However,



■ **FIGURE 6.2** **Translation of a fluid element.**

because of the presence of velocity gradients, the element will generally be deformed and rotated as it moves. For example, consider the effect of a single velocity gradient, $\partial u/\partial x$, on a small cube having sides δx, δy, and δz. As is shown in Fig. 6.3*a*, if the x component of velocity of O and B is u, then at nearby points A and C the x component of the velocity can be expressed as $u + (\partial u/\partial x)\,\delta x$. This difference in velocity causes a "stretching" of the volume element by an amount $(\partial u/\partial x)(\delta x)(\delta t)$ during the short time interval δt in which line OA stretches to OA' and BC to BC' (Fig. 6.3*b*). The corresponding change in the original volume, $\delta\forall = \delta x\,\delta y\,\delta z$, would be

$$\text{Change in } \delta\forall = \left(\frac{\partial u}{\partial x}\,\delta x\right)(\delta y\,\delta z)(\delta t)$$

and the *rate* at which the volume $\delta\forall$ is changing *per unit volume* due to the gradient $\partial u/\partial x$ is

$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \lim_{\delta t\to 0}\left[\frac{(\partial u/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial x} \tag{6.8}$$

The rate of volume change per unit volume is related to the velocity gradients.

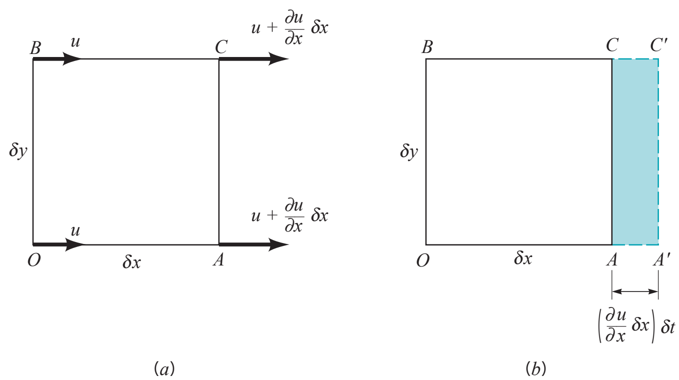
If velocity gradients $\partial v/\partial y$ and $\partial w/\partial z$ are also present, then using a similar analysis it follows that, in the general case,

$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \mathbf{V} \tag{6.9}$$

This rate of change of the volume per unit volume is called the ***volumetric dilatation rate.*** Thus, we see that the volume of a fluid may change as the element moves from one location to another in the flow field. However, for an *incompressible fluid* the volumetric dilatation rate is zero, since the element volume cannot change without a change in fluid density (the element mass must be conserved). Variations in the velocity in the direction of the velocity, as represented by the derivatives $\partial u/\partial x$, $\partial v/\partial y$, and $\partial w/\partial z$, simply cause a *linear deformation* of the element in the sense that the shape of the element does not change. Cross derivatives, such as $\partial u/\partial y$ and $\partial v/\partial x$, will cause the element to rotate and generally to undergo an *angular deformation*, which changes the shape of the element.

6.1.3 Angular Motion and Deformation

For simplicity we will consider motion in the x–y plane, but the results can be readily extended to the more general three dimensional case. The velocity variation that causes rotation and angular deformation is illustrated in Fig. 6.4*a*. In a short time interval δt the line segments OA and OB will



■ **FIGURE 6.3** **Linear deformation of a fluid element.**

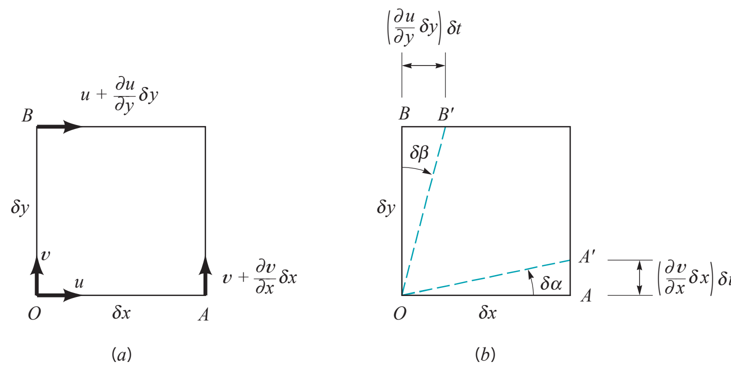


FIGURE 6.4 Angular motion and deformation of a fluid element.

V6.3 Shear deformation

rotate through the angles $\delta\alpha$ and $\delta\beta$ to the new positions OA' and OB', as is shown in Fig. 6.4*b*. The angular velocity of line OA, ω_{OA}, is

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t}$$

For small angles

$$\tan \delta\alpha \approx \delta\alpha = \frac{(\partial v/\partial x)\, \delta x\, \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t \tag{6.10}$$

so that

$$\omega_{OA} = \lim_{\delta t \to 0}\left[\frac{(\partial v/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial v}{\partial x}$$

Note that if $\partial v/\partial x$ is positive, ω_{OA} will be counterclockwise. Similarly, the angular velocity of the line OB is

$$\omega_{OB} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t}$$

and

$$\tan \delta\beta \approx \delta\beta = \frac{(\partial u/\partial y)\, \delta y\, \delta t}{\delta y} = \frac{\partial u}{\partial y}\delta t \tag{6.11}$$

so that

$$\omega_{OB} = \lim_{\delta t \to 0}\left[\frac{(\partial u/\partial y)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial y}$$

Rotation of fluid particles is related to certain velocity gradients in the flow field.

In this instance if $\partial u/\partial y$ is positive, ω_{OB} will be clockwise. The *rotation*, ω_z, of the element about the z axis is defined as the average of the angular velocities ω_{OA} and ω_{OB} of the two mutually perpendicular lines OA and OB.[1] Thus, if counterclockwise rotation is considered to be positive, it follows that

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \tag{6.12}$$

Rotation of the field element about the other two coordinate axes can be obtained in a similar manner with the result that for rotation about the x axis

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \tag{6.13}$$

and for rotation about the y axis

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \tag{6.14}$$

[1] With this definition ω_z can also be interpreted to be the angular velocity of the bisector of the angle between the lines OA and OB.

The three components, ω_x, ω_y, and ω_z can be combined to give the rotation vector, $\boldsymbol{\omega}$, in the form

$$\boldsymbol{\omega} = \omega_x \hat{\mathbf{i}} + \omega_y \hat{\mathbf{j}} + \omega_z \hat{\mathbf{k}} \tag{6.15}$$

An examination of this result reveals that $\boldsymbol{\omega}$ is equal to one-half the curl of the velocity vector. That is,

$$\boldsymbol{\omega} = \tfrac{1}{2} \text{ curl } \mathbf{V} = \tfrac{1}{2} \boldsymbol{\nabla} \times \mathbf{V} \tag{6.16}$$

since by definition of the vector operator $\boldsymbol{\nabla} \times \mathbf{V}$

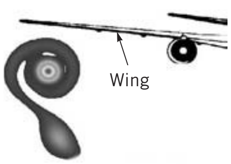
$$\frac{1}{2} \boldsymbol{\nabla} \times \mathbf{V} = \frac{1}{2} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\mathbf{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\mathbf{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}}$$

Vorticity in a flow field is related to fluid particle rotation.

The ***vorticity***, $\boldsymbol{\zeta}$, is defined as a vector that is twice the rotation vector; that is,

$$\boldsymbol{\zeta} = 2\,\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{V} \tag{6.17}$$



The use of the vorticity to describe the rotational characteristics of the fluid simply eliminates the $(\frac{1}{2})$ factor associated with the rotation vector. The figure in the margin shows vorticity contours of the wing tip vortex flow shortly after an aircraft has passed. The lighter colors indicate stronger vorticity. (See also Fig. 4.3.)

We observe from Eq. 6.12 that the fluid element will rotate about the z axis as an *undeformed* block (i.e., $\omega_{OA} = -\omega_{OB}$) only when $\partial u/\partial y = -\partial v/\partial x$. Otherwise the rotation will be associated with an angular deformation. We also note from Eq. 6.12 that when $\partial u/\partial y = \partial v/\partial x$ the rotation around the z axis is zero. More generally if $\boldsymbol{\nabla} \times \mathbf{V} = 0$, then the rotation (and the vorticity) are zero, and flow fields for which this condition applies are termed ***irrotational.*** We will find in Section 6.4 that the condition of irrotationality often greatly simplifies the analysis of complex flow fields. However, it is probably not immediately obvious why some flow fields would be irrotational, and we will need to examine this concept more fully in Section 6.4.

EXAMPLE 6.1 Vorticity

GIVEN For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

FIND Is this flow irrotational?

SOLUTION

For an irrotational flow the rotation vector, $\boldsymbol{\omega}$, having the components given by Eqs. 6.12, 6.13, and 6.14 must be zero. For the prescribed velocity field

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

and therefore

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

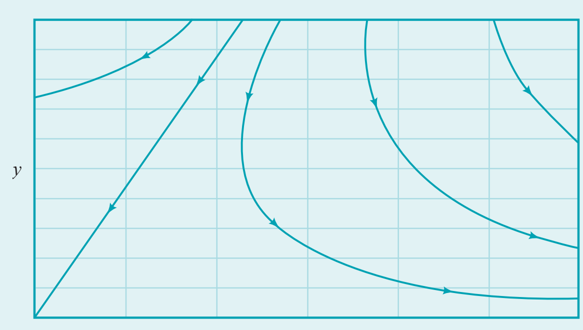
$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[(-2y) - (-2y)] = 0$$

Thus, the flow is irrotational. **(Ans)**

COMMENTS It is to be noted that for a two-dimensional flow field (where the flow is in the x–y plane) ω_x and ω_y will always be zero, since by definition of two-dimensional flow u and v are not functions of z, and w is zero. In this instance the condition for irrotationality simply becomes $\omega_z = 0$ or $\partial v/\partial x = \partial u/\partial y$.

The streamlines for the steady, two-dimensional flow of this example are shown in Fig. E6.1. (Information about how to calculate



■ FIGURE E6.1

streamlines for a given velocity field is given in Sections 4.1.4 and 6.2.3.) It is noted that all of the streamlines (except for the one through the origin) are curved. However, because the flow is irrotational, there is no rotation of the fluid elements. That is, lines *OA* and *OB* of Fig. 6.4 rotate with the same speed but in opposite directions.

As shown by Eq. 6.17, the condition of irrotationality is equivalent to the fact that the vorticity, $\boldsymbol{\zeta}$, is zero or the curl of the velocity is zero.

In addition to the rotation associated with the derivatives $\partial u/\partial y$ and $\partial v/\partial x$, it is observed from Fig. 6.4*b* that these derivatives can cause the fluid element to undergo an *angular deformation*, which results in a change in shape of the element. The change in the original right angle formed by the lines *OA* and *OB* is termed the shearing strain, $\delta\gamma$, and from Fig. 6.4*b*

$$\delta\gamma = \delta\alpha + \delta\beta$$

where $\delta\gamma$ is considered to be positive if the original right angle is decreasing. The rate of change of $\delta\gamma$ is called the *rate of shearing strain* or the *rate of angular deformation* and is commonly denoted with the symbol $\dot{\gamma}$. The angles $\delta\alpha$ and $\delta\beta$ are related to the velocity gradients through Eqs. 6.10 and 6.11 so that

$$\dot{\gamma} = \lim_{\delta t \to 0} \frac{\delta\gamma}{\delta t} = \lim_{\delta t \to 0} \left[\frac{(\partial v/\partial x)\,\delta t + (\partial u/\partial y)\,\delta t}{\delta t} \right]$$

and, therefore,

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{6.18}$$

As we will learn in Section 6.8, the rate of angular deformation is related to a corresponding shearing stress which causes the fluid element to change in shape. From Eq. 6.18 we note that if $\partial u/\partial y = -\partial v/\partial x$, the rate of angular deformation is zero, and this condition corresponds to the case in which the element is simply rotating as an undeformed block (Eq. 6.12). In the remainder of this chapter we will see how the various kinematical relationships developed in this section play an important role in the development and subsequent analysis of the differential equations that govern fluid motion.

6.2 Conservation of Mass

Conservation of mass requires that the mass of a system remain constant.

As is discussed in Section 5.1, conservation of mass requires that the mass, M, of a system remain constant as the system moves through the flow field. In equation form this principle is expressed as

$$\frac{DM_{\text{sys}}}{Dt} = 0$$

We found it convenient to use the control volume approach for fluid flow problems, with the control volume representation of the conservation of mass written as

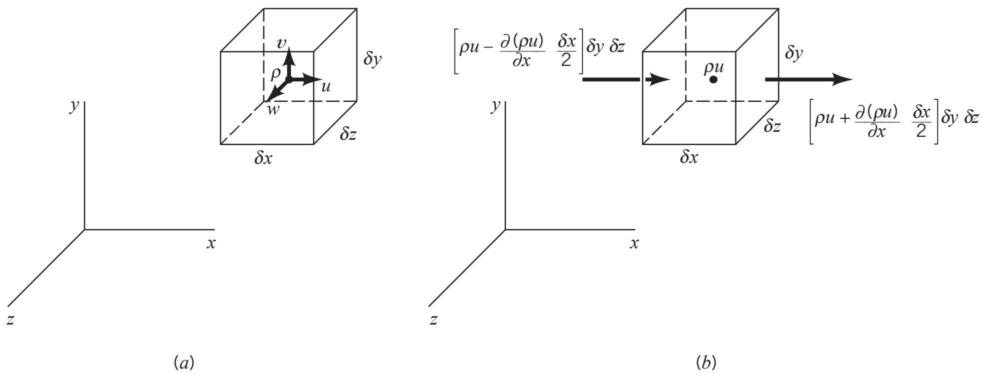
$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho\, d\forall + \int_{\text{cs}} \rho\, \mathbf{V} \cdot \hat{\mathbf{n}}\, dA = 0 \tag{6.19}$$

where the equation (commonly called the **continuity equation**) can be applied to a finite control volume (cv), which is bounded by a control surface (cs). The first integral on the left side of Eq. 6.19 represents the rate at which the mass within the control volume is changing, and the second integral represents the net rate at which mass is flowing out through the control surface (rate of mass outflow $-$ rate of mass inflow). To obtain the differential form of the continuity equation, Eq. 6.19 is applied to an infinitesimal control volume.

6.2.1 Differential Form of Continuity Equation

We will take as our control volume the small, stationary cubical element shown in Fig. 6.5*a*. At the center of the element the fluid density is ρ and the velocity has components u, v, and w. Since the element is small, the volume integral in Eq. 6.19 can be expressed as

$$\frac{\partial}{\partial t}\int_{\text{cv}} \rho\, d\forall \approx \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z \tag{6.20}$$



■ **FIGURE 6.5** **A differential element for the development of conservation of mass equation.**

The rate of mass flow through the surfaces of the element can be obtained by considering the flow in each of the coordinate directions separately. For example, in Fig. 6.5*b* flow in the *x* direction is depicted. If we let ρu represent the *x* component of the mass rate of flow per unit area at the center of the element, then on the right face

$$\rho u|_{x+(\delta x/2)} = \rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \tag{6.21}$$

and on the left face

$$\rho u|_{x-(\delta x/2)} = \rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2} \tag{6.22}$$

Note that we are really using a Taylor series expansion of ρu and neglecting higher order terms such as $(\delta x)^2$, $(\delta x)^3$, and so on. When the right-hand sides of Eqs. 6.21 and 6.22 are multiplied by the area $\delta y\,\delta z$, the rate at which mass is crossing the right and left sides of the element are obtained as is illustrated in Fig. 6.5*b*. When these two expressions are combined, the net rate of mass flowing from the element through the two surfaces can be expressed as

$$\begin{aligned}\begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } x \text{ direction}\end{array} &= \left[\rho u + \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z\\ &\quad - \left[\rho u - \frac{\partial(\rho u)}{\partial x}\frac{\delta x}{2}\right]\delta y\,\delta z = \frac{\partial(\rho u)}{\partial x}\,\delta x\,\delta y\,\delta z\end{aligned} \tag{6.23}$$

For simplicity, only flow in the *x* direction has been considered in Fig. 6.5*b*, but, in general, there will also be flow in the *y* and *z* directions. An analysis similar to the one used for flow in the *x* direction shows that

$$\begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } y \text{ direction}\end{array} = \frac{\partial(\rho v)}{\partial y}\,\delta x\,\delta y\,\delta z \tag{6.24}$$

and

$$\begin{array}{c}\text{Net rate of mass}\\ \text{outflow in } z \text{ direction}\end{array} = \frac{\partial(\rho w)}{\partial z}\,\delta x\,\delta y\,\delta z \tag{6.25}$$

Thus,

$$\begin{array}{c}\text{Net rate of}\\ \text{mass outflow}\end{array} = \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right]\delta x\,\delta y\,\delta z \tag{6.26}$$

From Eqs. 6.19, 6.20, and 6.26 it now follows that the differential equation for conservation of mass is

The continuity equation is one of the fundamental equations of fluid mechanics.

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0} \tag{6.27}$$

As previously mentioned, this equation is also commonly referred to as the continuity equation.

The continuity equation is one of the fundamental equations of fluid mechanics and, as expressed in Eq. 6.27, is valid for steady or unsteady flow, and compressible or incompressible fluids. In vector notation, Eq. 6.27 can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \tag{6.28}$$

Two special cases are of particular interest. For *steady* flow of *compressible* fluids

$$\nabla \cdot \rho \mathbf{V} = 0$$

or

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \tag{6.29}$$

For incompressible fluids the continuity equation reduces to a simple relationship involving certain velocity gradients.

This follows since by definition ρ is not a function of time for steady flow, but could be a function of position. For *incompressible* fluids the fluid density, ρ, is a constant throughout the flow field so that Eq. 6.28 becomes

$$\nabla \cdot \mathbf{V} = 0 \tag{6.30}$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6.31}$$

Equation 6.31 applies to both steady and unsteady flow of incompressible fluids. Note that Eq. 6.31 is the same as that obtained by setting the volumetric dilatation rate (Eq. 6.9) equal to zero. This result should not be surprising since both relationships are based on conservation of mass for incompressible fluids. However, the expression for the volumetric dilatation rate was developed from a system approach, whereas Eq. 6.31 was developed from a control volume approach. In the former case the deformation of a particular differential mass of fluid was studied, and in the latter case mass flow through a fixed differential volume was studied.

EXAMPLE 6.2 Continuity Equation

GIVEN The velocity components for a certain incompressible, steady flow field are

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z$$
$$w = ?$$

FIND Determine the form of the z component, w, required to satisfy the continuity equation.

SOLUTION

Any physically possible velocity distribution must for an incompressible fluid satisfy conservation of mass as expressed by the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

For the given velocity distribution

$$\frac{\partial u}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial v}{\partial y} = x + z$$

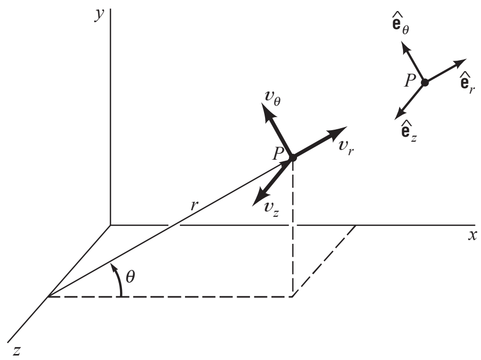
so that the required expression for $\partial w/\partial z$ is

$$\frac{\partial w}{\partial z} = -2x - (x + z) = -3x - z$$

Integration with respect to z yields

$$w = -3xz - \frac{z^2}{2} + f(x, y) \qquad \textbf{(Ans)}$$

COMMENT The third velocity component cannot be explicitly determined since the function $f(x, y)$ can have any form and conservation of mass will still be satisfied. The specific form of this function will be governed by the flow field described by these velocity components—that is, some additional information is needed to completely determine w.



■ **FIGURE 6.6** **The representation of velocity components in cylindrical polar coordinates.**

6.2.2 Cylindrical Polar Coordinates

For some problems, velocity components expressed in cylindrical polar coordinates will be convenient.

For some problems it is more convenient to express the various differential relationships in cylindrical polar coordinates rather than Cartesian coordinates. As is shown in Fig. 6.6, with cylindrical coordinates a point is located by specifying the coordinates r, θ, and z. The coordinate r is the radial distance from the z axis, θ is the angle measured from a line parallel to the x axis (with counterclockwise taken as positive), and z is the coordinate along the z axis. The velocity components, as sketched in Fig. 6.6, are the radial velocity, v_r, the tangential velocity, v_θ, and the axial velocity, v_z. Thus, the velocity at some arbitrary point P can be expressed as

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z \qquad \textbf{(6.32)}$$

where $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$, and $\hat{\mathbf{e}}_z$ are the unit vectors in the r, θ, and z directions, respectively, as are illustrated in Fig. 6.6. The use of cylindrical coordinates is particularly convenient when the boundaries of the flow system are cylindrical. Several examples illustrating the use of cylindrical coordinates will be given in succeeding sections in this chapter.

The differential form of the continuity equation in cylindrical coordinates is

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0} \qquad \textbf{(6.33)}$$

This equation can be derived by following the same procedure used in the preceding section (see Problem 6.20). For steady, compressible flow

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \qquad \textbf{(6.34)}$$

For incompressible fluids (for steady or unsteady flow)

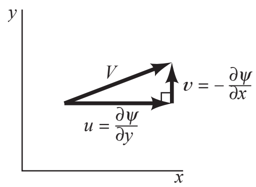
$$\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \qquad \textbf{(6.35)}$$

6.2.3 The Stream Function

Steady, incompressible, plane, two-dimensional flow represents one of the simplest types of flow of practical importance. By plane, two-dimensional flow we mean that there are only two velocity components, such as u and v, when the flow is considered to be in the x–y plane. For this flow the continuity equation, Eq. 6.31, reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \textbf{(6.36)}$$

Velocity components in a two-dimensional flow field can be expressed in terms of a stream function.



We still have two variables, u and v, to deal with, but they must be related in a special way as indicated by Eq. 6.36. This equation suggests that if we define a function $\psi(x, y)$, called the ***stream function,*** which relates the velocities shown by the figure in the margin as

$$\boxed{u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}} \tag{6.37}$$

then the continuity equation is identically satisfied. This conclusion can be verified by simply substituting the expressions for u and v into Eq. 6.36 so that

$$\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial x}\right) = \frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial^2\psi}{\partial y\,\partial x} = 0$$

Thus, whenever the velocity components are defined in terms of the stream function we know that conservation of mass will be satisfied. Of course, we still do not know what $\psi(x, y)$ is for a particular problem, but at least we have simplified the analysis by having to determine only one unknown function, $\psi(x, y)$, rather than the two functions, $u(x, y)$ and $v(x, y)$.

Another particular advantage of using the stream function is related to the fact that *lines along which ψ is constant are streamlines*. Recall from Section 4.1.4 that streamlines are lines in the flow field that are everywhere tangent to the velocities, as is illustrated in Fig. 6.7. It follows from the definition of the streamline that the slope at any point along a streamline is given by

$$\frac{dy}{dx} = \frac{v}{u}$$

The change in the value of ψ as we move from one point (x, y) to a nearby point $(x + dx, y + dy)$ is given by the relationship:

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = -v\,dx + u\,dy$$

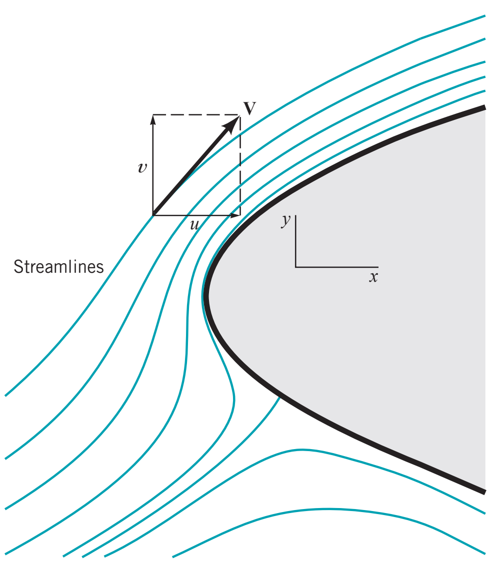
Along a line of constant ψ we have $d\psi = 0$ so that

$$-v\,dx + u\,dy = 0$$

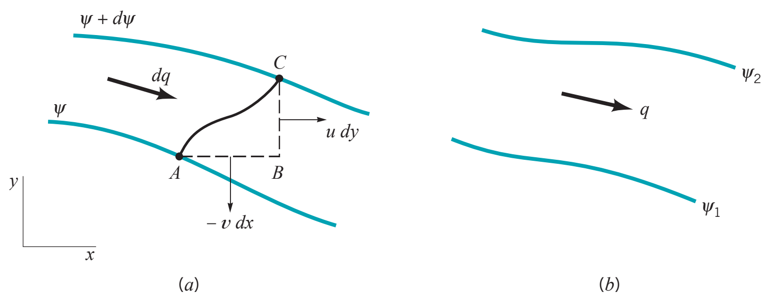
and, therefore, along a line of constant ψ

$$\frac{dy}{dx} = \frac{v}{u}$$

which is the defining equation for a streamline. Thus, if we know the function $\psi(x, y)$ we can plot lines of constant ψ to provide the family of streamlines that are helpful in visualizing the pattern



■ **FIGURE 6.7** **Velocity and velocity components along a streamline.**



■ **FIGURE 6.8** **The flow between two streamlines.**

of flow. There are an infinite number of streamlines that make up a particular flow field, since for each constant value assigned to ψ a streamline can be drawn.

The change in the value of the stream function is related to the volume rate of flow.

The actual numerical value associated with a particular streamline is not of particular significance, but the change in the value of ψ is related to the volume rate of flow. Consider two closely spaced streamlines, shown in Fig. 6.8*a*. The lower streamline is designated ψ and the upper one $\psi + d\psi$. Let dq represent the volume rate of flow (per unit width perpendicular to the x–y plane) passing between the two streamlines. Note that flow never crosses streamlines, since by definition the velocity is tangent to the streamline. From conservation of mass we know that the inflow, dq, crossing the arbitrary surface AC of Fig. 6.8*a* must equal the net outflow through surfaces AB and BC. Thus,

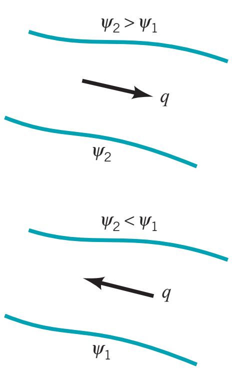
$$dq = u\,dy - v\,dx$$

or in terms of the stream function

$$dq = \frac{\partial \psi}{\partial y}\,dy + \frac{\partial \psi}{\partial x}\,dx \tag{6.38}$$

The right-hand side of Eq. 6.38 is equal to $d\psi$ so that

$$dq = d\psi \tag{6.39}$$



Thus, the volume rate of flow, q, between two streamlines such as ψ_1 and ψ_2 of Fig. 6.8*b* can be determined by integrating Eq. 6.39 to yield

$$q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1 \tag{6.40}$$

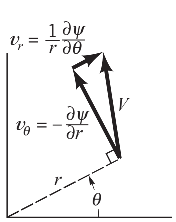
The relative value of ψ_2 with respect to ψ_1 determines the direction of flow, as shown by the figure in the margin.

In cylindrical coordinates the continuity equation (Eq. 6.35) for incompressible, plane, two-dimensional flow reduces to

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0 \tag{6.41}$$

and the velocity components, v_r and v_θ, can be related to the stream function, $\psi(r, \theta)$, through the equations

$$\boxed{v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}} \tag{6.42}$$



as shown by the figure in the margin.

Substitution of these expressions for the velocity components into Eq. 6.41 shows that the continuity equation is identically satisfied. The stream function concept can be extended to axisymmetric flows, such as flow in pipes or flow around bodies of revolution, and to two-dimensional compressible flows. However, the concept is not applicable to general three-dimensional flows.

EXAMPLE 6.3 Stream Function

GIVEN The velocity components in a steady, incompressible, two-dimensional flow field are

$$u = 2y$$
$$v = 4x$$

FIND
(a) Determine the corresponding stream function and
(b) Show on a sketch several streamlines. Indicate the direction of flow along the streamlines.

SOLUTION

(a) From the definition of the stream function (Eqs. 6.37)

$$u = \frac{\partial \psi}{\partial y} = 2y$$

and

$$v = -\frac{\partial \psi}{\partial x} = 4x$$

The first of these equations can be integrated to give

$$\psi = y^2 + f_1(x)$$

where $f_1(x)$ is an arbitrary function of x. Similarly from the second equation

$$\psi = -2x^2 + f_2(y)$$

where $f_2(y)$ is an arbitrary function of y. It now follows that in order to satisfy both expressions for the stream function

$$\psi = -2x^2 + y^2 + C \qquad \textbf{(Ans)}$$

where C is an arbitrary constant.

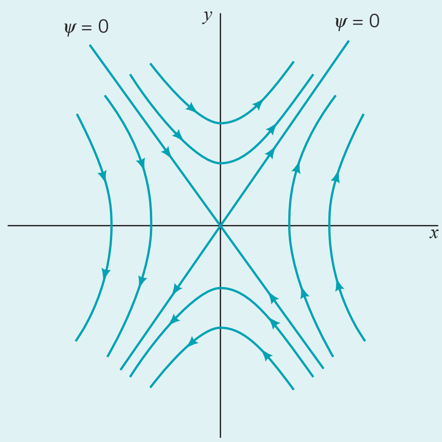
COMMENT Since the velocities are related to the derivatives of the stream function, an arbitrary constant can always be added to the function, and the value of the constant is actually of no consequence. Usually, for simplicity, we set $C = 0$ so that for this particular example the simplest form for the stream function is

$$\psi = -2x^2 + y^2 \qquad \textbf{(1)} \quad \textbf{(Ans)}$$

Either answer indicated would be acceptable.

(b) Streamlines can now be determined by setting $\psi =$ constant and plotting the resulting curve. With the above expression for ψ (with $C = 0$) the value of ψ at the origin is zero so that the equation of the streamline passing through the origin (the $\psi = 0$ streamline) is

$$0 = -2x^2 + y^2$$



■ FIGURE E6.3

or

$$y = \pm\sqrt{2}x$$

Other streamlines can be obtained by setting ψ equal to various constants. It follows from Eq. 1 that the equations of these streamlines (for $\psi \neq 0$) can be expressed in the form

$$\frac{y^2}{\psi} - \frac{x^2}{\psi/2} = 1$$

which we recognize as the equation of a hyperbola. Thus, the streamlines are a family of hyperbolas with the $\psi = 0$ streamlines as asymptotes. Several of the streamlines are plotted in Fig. E6.3. Since the velocities can be calculated at any point, the direction of flow along a given streamline can be easily deduced. For example, $v = -\partial\psi/\partial x = 4x$ so that $v > 0$ if $x > 0$ and $v < 0$ if $x < 0$. The direction of flow is indicated on the figure.

6.3 Conservation of Linear Momentum

To develop the differential momentum equations we can start with the linear momentum equation

$$\mathbf{F} = \left.\frac{D\mathbf{P}}{Dt}\right|_{\text{sys}} \tag{6.43}$$

where $\mathbf{F}$ is the resultant force acting on a fluid mass, $\mathbf{P}$ is the linear momentum defined as

$$\mathbf{P} = \int_{\text{sys}} \mathbf{V}\, dm$$

and the operator $D(\ )/Dt$ is the material derivative (see Section 4.2.12). In the last chapter it was demonstrated how Eq. 6.43 in the form

$$\sum \mathbf{F}_{\substack{\text{contents of the}\\ \text{control volume}}} = \frac{\partial}{\partial t}\int_{\text{cv}} \mathbf{V}\rho\, d\mathcal{V} + \int_{\text{cs}} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}\, dA \tag{6.44}$$

could be applied to a finite control volume to solve a variety of flow problems. To obtain the differential form of the linear momentum equation, we can either apply Eq. 6.43 to a differential system, consisting of a mass, δm, or apply Eq. 6.44 to an infinitesimal control volume, $\delta\mathcal{V}$, which initially bounds the mass δm. It is probably simpler to use the system approach since application of Eq. 6.43 to the differential mass, δm, yields

$$\delta\mathbf{F} = \frac{D(\mathbf{V}\,\delta m)}{Dt}$$

where $\delta\mathbf{F}$ is the resultant force acting on δm. Using this system approach δm can be treated as a constant so that

$$\delta\mathbf{F} = \delta m\,\frac{D\mathbf{V}}{Dt}$$

But $D\mathbf{V}/Dt$ is the acceleration, $\mathbf{a}$, of the element. Thus,

$$\delta\mathbf{F} = \delta m\,\mathbf{a} \tag{6.45}$$

which is simply Newton's second law applied to the mass δ_m. This is the same result that would be obtained by applying Eq. 6.44 to an infinitesimal control volume (see Ref. 1). Before we can proceed, it is necessary to examine how the force $\delta\mathbf{F}$ can be most conveniently expressed.

6.3.1 Description of Forces Acting on the Differential Element

Both surface forces and body forces generally act on fluid particles.

In general, two types of forces need to be considered: *surface forces*, which act on the surface of the differential element, and *body forces*, which are distributed throughout the element. For our purpose, the only body force, $\delta\mathbf{F}_b$, of interest is the weight of the element, which can be expressed as

$$\delta\mathbf{F}_b = \delta m\,\mathbf{g} \tag{6.46}$$

where $\mathbf{g}$ is the vector representation of the acceleration of gravity. In component form

$$\delta F_{bx} = \delta m\, g_x \tag{6.47a}$$

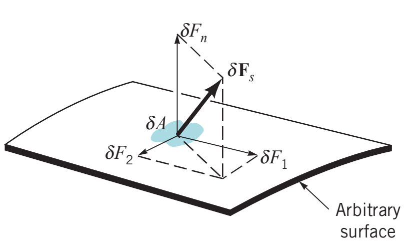
$$\delta F_{by} = \delta m\, g_y \tag{6.47b}$$

$$\delta F_{bz} = \delta m\, g_z \tag{6.47c}$$

where g_x, g_y, and g_z are the components of the acceleration of gravity vector in the x, y, and z directions, respectively.

Surface forces act on the element as a result of its interaction with its surroundings. At any arbitrary location within a fluid mass, the force acting on a small area, δA, which lies in an arbitrary surface, can be represented by $\delta\mathbf{F}_s$, as is shown in Fig. 6.9. In general, $\delta\mathbf{F}_s$ will be inclined with respect to the surface. The force $\delta\mathbf{F}_s$ can be resolved into three components, δF_n, δF_1, and δF_2, where δF_n is normal to the area, δA, and δF_1 and δF_2 are parallel to the area and orthogonal to each other. The *normal stress*, σ_n, is defined as

$$\sigma_n = \lim_{\delta A\to 0}\frac{\delta F_n}{\delta A}$$



■ **FIGURE 6.9 Components of force acting on an arbitrary differential area.**

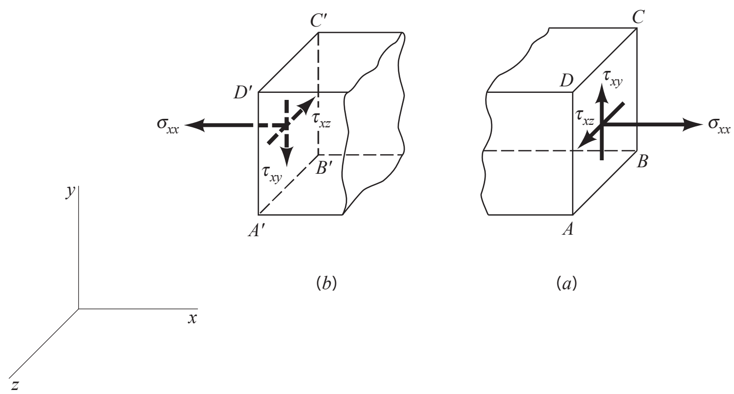


FIGURE 6.10 Double subscript notation for stresses.

and the *shearing stresses* are defined as

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$$

and

$$\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$

We will use σ for normal stresses and τ for shearing stresses. The intensity of the force per unit area at a point in a body can thus be characterized by a normal stress and two shearing stresses, if the orientation of the area is specified. For purposes of analysis it is usually convenient to reference the area to the coordinate system. For example, for the rectangular coordinate system shown in Fig. 6.10 we choose to consider the stresses acting on planes parallel to the coordinate planes. On the plane *ABCD* of Fig. 6.10*a*, which is parallel to the *y–z* plane, the normal stress is denoted σ_{xx} and the shearing stresses are denoted as τ_{xy} and τ_{xz}. To easily identify the particular stress component we use a double subscript notation. The first subscript indicates the direction of the *normal* to the plane on which the stress acts, and the second subscript indicates the direction of the stress. Thus, normal stresses have repeated subscripts, whereas the subscripts for the shearing stresses are always different.

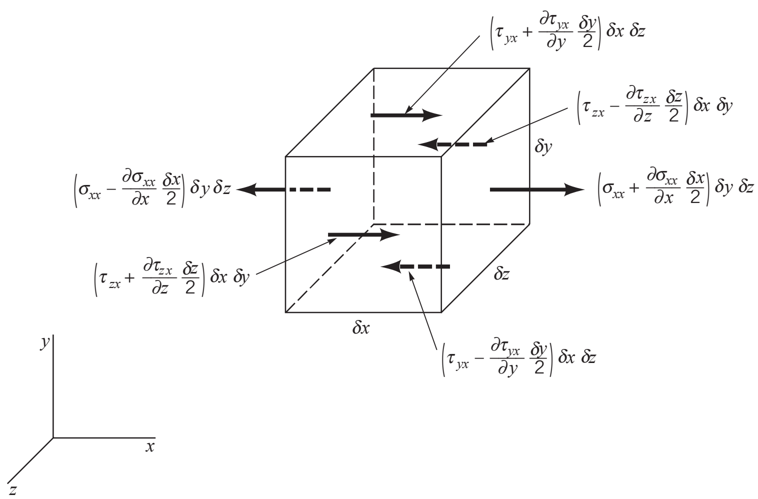
It is also necessary to establish a sign convention for the stresses. We define the positive direction for the stress as the positive coordinate direction on the surfaces for which the outward normal is in the positive coordinate direction. This is the case illustrated in Fig. 6.10*a* where the outward normal to the area *ABCD* is in the positive *x* direction. The positive directions for σ_{xx}, τ_{xy}, and τ_{xz} are as shown in Fig. 6.10*a*. If the outward normal points in the negative coordinate direction, as in Fig. 6.10*b* for the area $A'B'C'D'$, then the stresses are considered positive if directed in the negative coordinate directions. Thus, the stresses shown in Fig. 6.10*b* are considered to be positive when directed as shown. Note that positive normal stresses are tensile stresses; that is, they tend to "stretch" the material.

It should be emphasized that the state of stress at a point in a material is not completely defined by simply three components of a "stress vector." This follows, since any particular stress vector depends on the orientation of the plane passing through the point. However, it can be shown that the normal and shearing stresses acting on *any* plane passing through a point can be expressed in terms of the stresses acting on three orthogonal planes passing through the point (Ref. 2).

Surface forces can be expressed in terms of the shear and normal stresses.

We now can express the surface forces acting on a small cubical element of fluid in terms of the stresses acting on the faces of the element as shown in Fig. 6.11. It is expected that in general the stresses will vary from point to point within the flow field. Thus, through the use of Taylor series expansions we will express the stresses on the various faces in terms of the corresponding stresses at the center of the element of Fig. 6.11 and their gradients in the coordinate directions. For simplicity only the forces in the *x* direction are shown. Note that the stresses must be multiplied by the area on which they act to obtain the force. Summing all these forces in the *x* direction yields

$$\delta F_{sx} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \right) \delta x \, \delta y \, \delta z \qquad \textbf{(6.48a)}$$



■ **FIGURE 6.11** **Surface forces in the x direction acting on a fluid element.**

for the resultant surface force in the x direction. In a similar manner the resultant surface forces in the y and z directions can be obtained and expressed as

$$\delta F_{sy} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x\, \delta y\, \delta z \tag{6.48b}$$

$$\delta F_{sz} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\, \delta y\, \delta z \tag{6.48c}$$

The resultant surface force can now be expressed as

$$\delta \mathbf{F}_s = \delta F_{sx}\hat{\mathbf{i}} + \delta F_{sy}\hat{\mathbf{j}} + \delta F_{sz}\hat{\mathbf{k}} \tag{6.49}$$

and this force combined with the body force, $\delta \mathbf{F}_b$, yields the resultant force, $\delta \mathbf{F}$, acting on the differential mass, δm. That is, $\delta \mathbf{F} = \delta \mathbf{F}_s + \delta \mathbf{F}_b$.

6.3.2 Equations of Motion

The expressions for the body and surface forces can now be used in conjunction with Eq. 6.45 to develop the equations of motion. In component form Eq. 6.45 can be written as

$$\delta F_x = \delta m\, a_x$$

$$\delta F_y = \delta m\, a_y$$

$$\delta F_z = \delta m\, a_z$$

where $\delta m = \rho\, \delta x\, \delta y\, \delta z$, and the acceleration components are given by Eq. 6.3. It now follows (using Eqs. 6.47 and 6.48 for the forces on the element) that

The motion of a fluid is governed by a set of nonlinear differential equations.

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \tag{6.50a}$$

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \tag{6.50b}$$

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \tag{6.50c}$$

where the element volume $\delta x\, \delta y\, \delta z$ cancels out.

Equations 6.50 are the general differential equations of motion for a fluid. In fact, they are applicable to any continuum (solid or fluid) in motion or at rest. However, before we can use the equations to solve specific problems, some additional information about the stresses must be obtained.

Otherwise, we will have more unknowns (all of the stresses and velocities and the density) than equations. It should not be too surprising that the differential analysis of fluid motion is complicated. We are attempting to describe, in detail, complex fluid motion.

6.4 Inviscid Flow

As is discussed in Section 1.6, shearing stresses develop in a moving fluid because of the viscosity of the fluid. We know that for some common fluids, such as air and water, the viscosity is small, and therefore it seems reasonable to assume that under some circumstances we may be able to simply neglect the effect of viscosity (and thus shearing stresses). Flow fields in which the shearing stresses are assumed to be negligible are said to be *inviscid*, *nonviscous*, or *frictionless*. These terms are used interchangeably. As is discussed in Section 2.1, for fluids in which there are no shearing stresses the normal stress at a point is independent of direction—that is, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz}$. In this instance we define the pressure, p, as the negative of the normal stress so that

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

The negative sign is used so that a *compressive* normal stress (which is what we expect in a fluid) will give a *positive* value for p.

In Chapter 3 the inviscid flow concept was used in the development of the Bernoulli equation, and numerous applications of this important equation were considered. In this section we will again consider the Bernoulli equation and will show how it can be derived from the general equations of motion for inviscid flow.

6.4.1 Euler's Equations of Motion

For an inviscid flow in which all the shearing stresses are zero, and the normal stresses are replaced by $-p$, the general equations of motion (Eqs. 6.50) reduce to

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) \qquad \textbf{(6.51a)}$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) \qquad \textbf{(6.51b)}$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) \qquad \textbf{(6.51c)}$$

These equations are commonly referred to as ***Euler's equations of motion,*** named in honor of Leonhard Euler (1707–1783), a famous Swiss mathematician who pioneered work on the relationship between pressure and flow. In vector notation Euler's equations can be expressed as

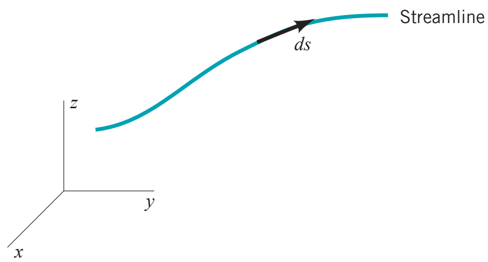
Euler's equations of motion apply to an inviscid flow field.

$$\rho\mathbf{g} - \nabla p = \rho\left[\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right] \qquad \textbf{(6.52)}$$

Although Eqs. 6.51 are considerably simpler than the general equations of motion, Eqs. 6.50, they are still not amenable to a general analytical solution that would allow us to determine the pressure and velocity at all points within an inviscid flow field. The main difficulty arises from the nonlinear velocity terms ($u\,\partial u/\partial x$, $v\,\partial u/\partial y$, etc.), which appear in the convective acceleration. Because of these terms, Euler's equations are nonlinear partial differential equations for which we do not have a general method of solving. However, under some circumstances we can use them to obtain useful information about inviscid flow fields. For example, as shown in the following section we can integrate Eq. 6.52 to obtain a relationship (the Bernoulli equation) between elevation, pressure, and velocity along a streamline.

6.4.2 The Bernoulli Equation

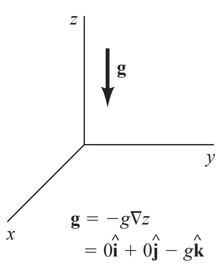
In Section 3.2 the Bernoulli equation was derived by a direct application of Newton's second law to a fluid particle moving along a streamline. In this section we will again derive this important



■ FIGURE 6.12 The notation for differential length along a streamline.

equation, starting from Euler's equations. Of course, we should obtain the same result since Euler's equations simply represent a statement of Newton's second law expressed in a general form that is useful for flow problems and maintains the restriction of zero viscosity. We will restrict our attention to steady flow so Euler's equation in vector form becomes

$$\rho \mathbf{g} - \boldsymbol{\nabla} p = \rho(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \tag{6.53}$$



We wish to integrate this differential equation along some arbitrary streamline (Fig. 6.12) and select the coordinate system with the z axis vertical (with "up" being positive) so that, as shown by the figure in the margin, the acceleration of gravity vector can be expressed as

$$\mathbf{g} = -g\boldsymbol{\nabla} z$$

where g is the magnitude of the acceleration of gravity vector. Also, it will be convenient to use the vector identity

$$(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} = \tfrac{1}{2}\boldsymbol{\nabla}(\mathbf{V} \cdot \mathbf{V}) - \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})$$

Equation 6.53 can now be written in the form

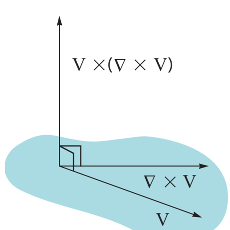
$$-\rho g \boldsymbol{\nabla} z - \boldsymbol{\nabla} p = \frac{\rho}{2}\boldsymbol{\nabla}(\mathbf{V} \cdot \mathbf{V}) - \rho \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})$$

and this equation can be rearranged to yield

$$\frac{\boldsymbol{\nabla} p}{\rho} + \frac{1}{2}\boldsymbol{\nabla}(V^2) + g\boldsymbol{\nabla} z = \mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})$$

We next take the dot product of each term with a differential length $d\mathbf{s}$ along a streamline (Fig. 6.12). Thus,

$$\frac{\boldsymbol{\nabla} p}{\rho} \cdot d\mathbf{s} + \frac{1}{2}\boldsymbol{\nabla}(V^2) \cdot d\mathbf{s} + g\boldsymbol{\nabla} z \cdot d\mathbf{s} = [\mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})] \cdot d\mathbf{s} \tag{6.54}$$



Since $d\mathbf{s}$ has a direction along the streamline, the vectors $d\mathbf{s}$ and $\mathbf{V}$ are parallel. However, as shown by the figure in the margin, the vector $\mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})$ is perpendicular to $\mathbf{V}$ (why?), so it follows that

$$[\mathbf{V} \times (\boldsymbol{\nabla} \times \mathbf{V})] \cdot d\mathbf{s} = 0$$

Recall also that the dot product of the gradient of a scalar and a differential length gives the differential change in the scalar in the direction of the differential length. That is, with $d\mathbf{s} = dx\,\hat{\mathbf{i}} + dy\,\hat{\mathbf{j}} + dz\,\hat{\mathbf{k}}$ we can write $\boldsymbol{\nabla} p \cdot d\mathbf{s} = (\partial p/\partial x)\,dx + (\partial p/\partial y)dy + (\partial p/\partial z)dz = dp$. Thus, Eq. 6.54 becomes

$$\frac{dp}{\rho} + \frac{1}{2}d(V^2) + g\,dz = 0 \tag{6.55}$$

Euler's equations can be arranged to give the relationship among pressure, velocity, and elevation for inviscid fluids

where the change in p, V, and z is along the streamline. Equation 6.55 can now be integrated to give

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{6.56}$$

which indicates that the sum of the three terms on the left side of the equation must remain a constant along a given streamline. Equation 6.56 is valid for both compressible and incompressible inviscid flows, but for compressible fluids the variation in ρ with p must be specified before the first term in Eq. 6.56 can be evaluated.

For inviscid, incompressible fluids (commonly called ***ideal fluids***) Eq. 6.56 can be written as

$$\boxed{\frac{p}{\rho} + \frac{V^2}{2} + gz = \text{constant along a streamline}} \tag{6.57}$$

and this equation is the ***Bernoulli equation*** used extensively in Chapter 3. It is often convenient to write Eq. 6.57 between two points (1) and (2) along a streamline and to express the equation in the "head" form by dividing each term by g so that

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2} \tag{6.58}$$

It should be again emphasized that the Bernoulli equation, as expressed by Eqs. 6.57 and 6.58, is restricted to the following:

- inviscid flow
- incompressible flow
- steady flow
- flow along a streamline

You may want to go back and review some of the examples in Chapter 3 that illustrate the use of the Bernoulli equation.

6.4.3 Irrotational Flow

The vorticity is zero in an irrotational flow field.

If we make one additional assumption—that the flow is *irrotational*—the analysis of inviscid flow problems is further simplified. Recall from Section 6.1.3 that the rotation of a fluid element is equal to $\frac{1}{2}(\nabla \times \mathbf{V})$, and an irrotational flow field is one for which $\nabla \times \mathbf{V} = 0$ (i.e., the curl of velocity is zero). Since the vorticity, ζ, is defined as $\nabla \times \mathbf{V}$, it also follows that in an irrotational flow field the vorticity is zero. The concept of irrotationality may seem to be a rather strange condition for a flow field. Why would a flow field be irrotational? To answer this question we note that if $\frac{1}{2}(\nabla \times \mathbf{V}) = 0$, then each of the components of this vector, as are given by Eqs. 6.12, 6.13, and 6.14, must be equal to zero. Since these components include the various velocity gradients in the flow field, the condition of irrotationality imposes specific relationships among these velocity gradients. For example, for rotation about the z axis to be zero, it follows from Eq. 6.12 that

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

and, therefore,

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \tag{6.59}$$

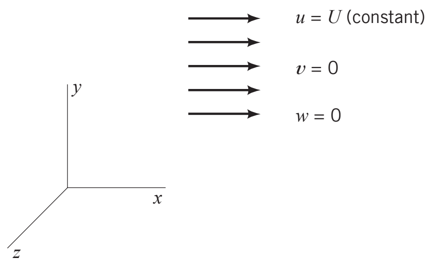
Similarly from Eqs. 6.13 and 6.14

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \tag{6.60}$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \tag{6.61}$$

A general flow field would not satisfy these three equations. However, a uniform flow as is illustrated in Fig. 6.13 does. Since $u = U$ (a constant), $v = 0$, and $w = 0$, it follows that Eqs. 6.59, 6.60, and 6.61 are all satisfied. Therefore, a uniform flow field (in which there are no velocity gradients) is certainly an example of an irrotational flow.

Uniform flows by themselves are not very interesting. However, many interesting and important flow problems include uniform flow in some part of the flow field. Two examples are

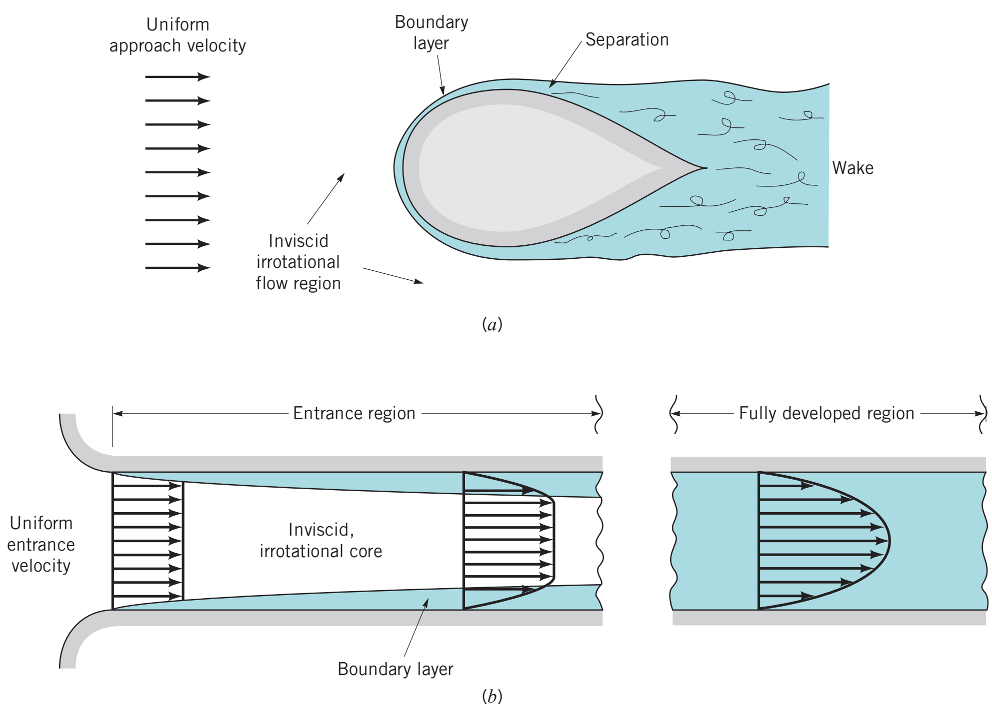


■ FIGURE 6.13 **Uniform flow in the x direction.**

shown in Fig. 6.14. In Fig. 6.14*a* a solid body is placed in a uniform stream of fluid. Far away from the body the flow remains uniform, and in this far region the flow is irrotational. In Fig. 6.14*b*, flow from a large reservoir enters a pipe through a streamlined entrance where the velocity distribution is essentially uniform. Thus, at the entrance the flow is irrotational.

Flow fields involving real fluids often include both regions of negligible shearing stresses and regions of significant shearing stresses.

For an inviscid fluid there are no shearing stresses—the only forces acting on a fluid element are its weight and pressure forces. Since the weight acts through the element center of gravity, and the pressure acts in a direction normal to the element surface, neither of these forces can cause the element to rotate. Therefore, for an inviscid fluid, if some part of the flow field is irrotational, the fluid elements emanating from this region will not take on any rotation as they progress through the flow field. This phenomenon is illustrated in Fig. 6.14*a* in which fluid elements flowing far away from the body have irrotational motion, and as they flow around the body the motion remains irrotational except very near the boundary. Near the boundary the velocity changes rapidly from zero at the boundary (no-slip condition) to some relatively large value in a short distance from the boundary. This rapid change in velocity gives rise to a large velocity gradient normal to the boundary and produces significant shearing stresses, even though the viscosity is small. Of course if we had a truly inviscid fluid, the fluid would simply "slide" past the boundary and the flow would be irrotational everywhere. But this is not the case for real fluids, so we will typically have a layer (usually very thin) near any fixed surface in a moving stream in which shearing stresses are not negligible. This layer is called the *boundary layer*. Outside the boundary layer the flow can be treated as an irrotational flow. Another possible consequence of the boundary layer is that the main stream may "separate" from the surface and form a *wake* downstream from the body. (See the



■ FIGURE 6.14 **Various regions of flow: (*a*) around bodies; (*b*) through channels.**

photographs at the beginning of Chapters 7, 9, and 11.) The wake would include a region of slow, perhaps randomly moving fluid. To completely analyze this type of problem it is necessary to consider both the inviscid, irrotational flow outside the boundary layer, and the viscous, rotational flow within the boundary layer and to somehow "match" these two regions. This type of analysis is considered in Chapter 9.

As is illustrated in Fig. 6.14*b*, the flow in the entrance to a pipe may be uniform (if the entrance is streamlined), and thus will be irrotational. In the central core of the pipe the flow remains irrotational for some distance. However, a boundary layer will develop along the wall and grow in thickness until it fills the pipe. Thus, for this type of internal flow there will be an *entrance region* in which there is a central irrotational core, followed by a so-called *fully developed region* in which viscous forces are dominant. The concept of irrotationality is completely invalid in the fully developed region. This type of internal flow problem is considered in detail in Chapter 8.

The two preceding examples are intended to illustrate the possible applicability of irrotational flow to some "real fluid" flow problems and to indicate some limitations of the irrotationality concept. We proceed to develop some useful equations based on the assumptions of inviscid, incompressible, irrotational flow, with the admonition to use caution when applying the equations.

6.4.4 The Bernoulli Equation for Irrotational Flow

In the development of the Bernoulli equation in Section 6.4.2, Eq. 6.54 was integrated along a streamline. This restriction was imposed so the right side of the equation could be set equal to zero; that is,

$$[\mathbf{V} \times (\nabla \times \mathbf{V})] \cdot d\mathbf{s} = 0$$

(since $d\mathbf{s}$ is parallel to $\mathbf{V}$). However, for irrotational flow, $\nabla \times \mathbf{V} = 0$, so the right side of Eq. 6.54 is zero regardless of the direction of $d\mathbf{s}$. We can now follow the same procedure used to obtain Eq. 6.55, where the differential changes dp, $d(V^2)$, and dz can be taken in any direction. Integration of Eq. 6.55 again yields

$$\int \frac{dp}{\rho} + \frac{V^2}{2} + gz = \text{constant} \tag{6.62}$$

where for irrotational flow the constant is the same throughout the flow field. Thus, for incompressible, irrotational flow the Bernoulli equation can be written as

$$\boxed{\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2} \tag{6.63}$$

The Bernoulli equation can be applied between any two points in an irrotational flow field.

between *any two points in the flow field*. Equation 6.63 is exactly the same form as Eq. 6.58 but is not limited to application along a streamline. However, Eq. 6.63 is restricted to

- inviscid flow
- steady flow
- incompressible flow
- irrotational flow

It may be worthwhile to review the use and misuse of the Bernoulli equation for rotational flow as is illustrated in Example 3.18.

6.4.5 The Velocity Potential

For an irrotational flow the velocity gradients are related through Eqs. 6.59, 6.60, and 6.61. It follows that in this case the velocity components can be expressed in terms of a scalar function $\phi(x, y, z, t)$ as

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z} \tag{6.64}$$

where ϕ is called the ***velocity potential.*** Direct substitution of these expressions for the velocity components into Eqs. 6.59, 6.60, and 6.61 will verify that a velocity field defined by Eqs. 6.64 is indeed irrotational. In vector form, Eqs. 6.64 can be written as

$$\boxed{\mathbf{V} = \boldsymbol{\nabla}\phi} \tag{6.65}$$

so that for an irrotational flow the velocity is expressible as the gradient of a scalar function ϕ.

The velocity potential is a consequence of the irrotationality of the flow field, whereas the stream function is a consequence of conservation of mass (see Section 6.2.3). It is to be noted, however, that the velocity potential can be defined for a general three-dimensional flow, whereas the stream function is restricted to two-dimensional flows.

For an incompressible fluid we know from conservation of mass that

$$\boldsymbol{\nabla} \cdot \mathbf{V} = 0$$

Inviscid, incompressible, irrotational flow fields are governed by Laplace's equation and are called potential flows.

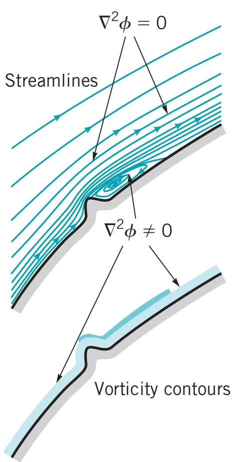
and therefore for incompressible, irrotational flow (with $\mathbf{V} = \boldsymbol{\nabla}\phi$) it follows that

$$\boldsymbol{\nabla}^2\phi = 0 \tag{6.66}$$

where $\boldsymbol{\nabla}^2(\;) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}(\;)$ is the *Laplacian operator*. In Cartesian coordinates

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

This differential equation arises in many different areas of engineering and physics and is called *Laplace's equation*. Thus, inviscid, incompressible, irrotational flow fields are governed by Laplace's equation. This type of flow is commonly called a ***potential flow.*** To complete the mathematical formulation of a given problem, boundary conditions have to be specified. These are usually velocities specified on the boundaries of the flow field of interest. It follows that if the potential function can be determined, then the velocity at all points in the flow field can be determined from Eq. 6.64, and the pressure at all points can be determined from the Bernoulli equation (Eq. 6.63). Although the concept of the velocity potential is applicable to both steady and unsteady flow, we will confine our attention to steady flow.



Potential flows, governed by Eqs. 6.64 and 6.66, are irrotational flows. That is, the vorticity is zero throughout. If vorticity is present (e.g., boundary layer, wake), then the flow cannot be described by Laplace's equation. The figure in the margin illustrates a flow in which the vorticity is not zero in two regions—the separated region behind the bump and the boundary layer next to the solid surface. This is discussed in detail in Chapter 9.

For some problems it will be convenient to use cylindrical coordinates, r, θ, and z. In this coordinate system the gradient operator is

$$\boldsymbol{\nabla}(\;) = \frac{\partial(\;)}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial(\;)}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\partial(\;)}{\partial z}\hat{\mathbf{e}}_z \tag{6.67}$$

so that

$$\boldsymbol{\nabla}\phi = \frac{\partial\phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\partial\phi}{\partial z}\hat{\mathbf{e}}_z \tag{6.68}$$

where $\phi = \phi(r, \theta, z)$. Since

$$\mathbf{V} = v_r\hat{\mathbf{e}}_r + v_\theta\hat{\mathbf{e}}_\theta + v_z\hat{\mathbf{e}}_z \tag{6.69}$$

it follows for an irrotational flow (with $\mathbf{V} = \boldsymbol{\nabla}\phi$)

$$v_r = \frac{\partial\phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \qquad v_z = \frac{\partial\phi}{\partial z} \tag{6.70}$$

Also, Laplace's equation in cylindrical coordinates is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \tag{6.71}$$

EXAMPLE 6.4 Velocity Potential and Inviscid Flow Pressure

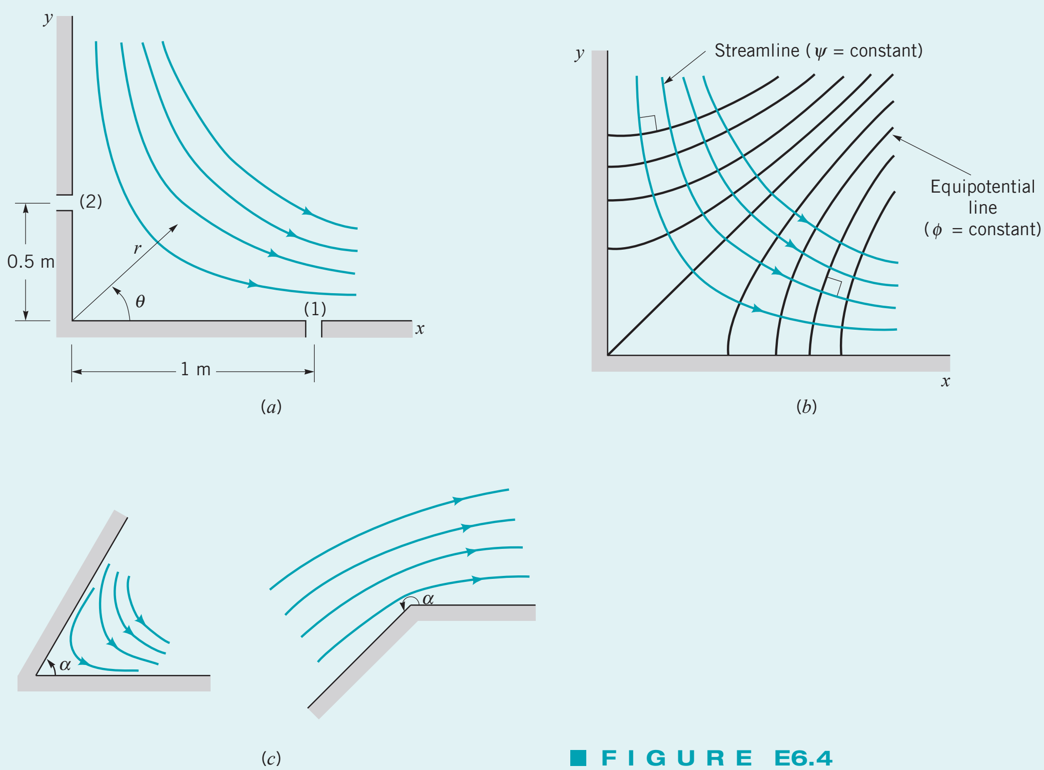
GIVEN The two-dimensional flow of a nonviscous, incompressible fluid in the vicinity of the 90° corner of Fig. E6.4*a* is described by the stream function

$$\psi = 2r^2 \sin 2\theta$$

where ψ has units of m^2/s when r is in meters. Assume the fluid density is 10^3 kg/m^3 and the x–y plane is horizontal—that is, there is no difference in elevation between points (1) and (2).

FIND

(a) Determine, if possible, the corresponding velocity potential.
(b) If the pressure at point (1) on the wall is 30 kPa, what is the pressure at point (2)?



■ FIGURE E6.4

SOLUTION

(a) The radial and tangential velocity components can be obtained from the stream function as (see Eq. 6.42)

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = 4r \cos 2\theta$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -4r \sin 2\theta$$

Since

$$v_r = \frac{\partial \phi}{\partial r}$$

it follows that

$$\frac{\partial \phi}{\partial r} = 4r \cos 2\theta$$

and therefore by integration

$$\phi = 2r^2 \cos 2\theta + f_1(\theta) \quad \textbf{(1)}$$

where $f_1(\theta)$ is an arbitrary function of θ. Similarly

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -4r \sin 2\theta$$

and integration yields

$$\phi = 2r^2 \cos 2\theta + f_2(r) \quad \textbf{(2)}$$

where $f_2(r)$ is an arbitrary function of r. To satisfy both Eqs. 1 and 2, the velocity potential must have the form

$$\phi = 2r^2 \cos 2\theta + C \quad \textbf{(Ans)}$$

where C is an arbitrary constant. As is the case for stream functions, the specific value of C is not important, and it is customary to let $C = 0$ so that the velocity potential for this corner flow is

$$\phi = 2r^2 \cos 2\theta \quad \textbf{(Ans)}$$

COMMENT In the statement of this problem it was implied by the wording "if possible" that we might not be able to find a corresponding velocity potential. The reason for this concern is that we can always define a stream function for two-dimensional flow, but the flow must be *irrotational* if there is a corresponding velocity potential. Thus, the fact that we were able to determine a velocity potential means that the flow is irrotational. Several streamlines and lines of constant ϕ are plotted in Fig. E6.4*b*. These two sets of lines are *orthogonal*. The reason why streamlines and lines of constant ϕ are always orthogonal is explained in Section 6.5.

(b) Since we have an irrotational flow of a nonviscous, incompressible fluid, the Bernoulli equation can be applied between any two points. Thus, between points (1) and (2) with no elevation change

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + \frac{V_2^2}{2g}$$

or

$$p_2 = p_1 + \frac{\rho}{2}(V_1^2 - V_2^2) \tag{3}$$

Since

$$V^2 = v_r^2 + v_\theta^2$$

it follows that for any point within the flow field

$$\begin{aligned} V^2 &= (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2 \\ &= 16r^2(\cos^2 2\theta + \sin^2 2\theta) \\ &= 16r^2 \end{aligned}$$

This result indicates that the square of the velocity at any point depends only on the radial distance, r, to the point. Note that the constant, 16, has units of s^{-2}. Thus,

$$V_1^2 = (16\ s^{-2})(1\ m)^2 = 16\ m^2/s^2$$

and

$$V_2^2 = (16\ s^{-2})(0.5\ m)^2 = 4\ m^2/s^2$$

Substitution of these velocities into Eq. 3 gives

$$\begin{aligned} p_2 &= 30 \times 10^3\ N/m^2 + \frac{10^3\ kg/m^3}{2}(16\ m^2/s^2 - 4\ m^2/s^2) \\ &= 36\ kPa \end{aligned}$$ **(Ans)**

COMMENT The stream function used in this example could also be expressed in Cartesian coordinates as

$$\psi = 2r^2 \sin 2\theta = 4r^2 \sin\theta\cos\theta$$

or

$$\psi = 4xy$$

since $x = r\cos\theta$ and $y = r\sin\theta$. However, in the cylindrical polar form the results can be generalized to describe flow in the vicinity of a corner of angle α (see Fig. E6.4*c*) with the equations

$$\psi = Ar^{\pi/\alpha}\sin\frac{\pi\theta}{\alpha}$$

and

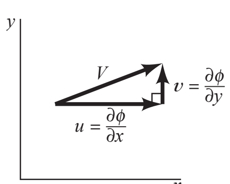
$$\phi = Ar^{\pi/\alpha}\cos\frac{\pi\theta}{\alpha}$$

where A is a constant.

6.5 Some Basic, Plane Potential Flows

For potential flow, basic solutions can be simply added to obtain more complicated solutions.

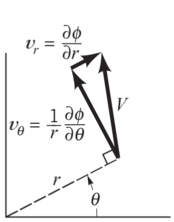
A major advantage of Laplace's equation is that it is a linear partial differential equation. Since it is linear, various solutions can be added to obtain other solutions—that is, if $\phi_1(x, y, z)$ and $\phi_2(x, y, z)$ are two solutions to Laplace's equation, then $\phi_3 = \phi_1 + \phi_2$ is also a solution. The practical implication of this result is that if we have certain basic solutions we can combine them to obtain more complicated and interesting solutions. In this section several basic velocity potentials, which describe some relatively simple flows, will be determined. In the next section these basic potentials will be combined to represent complicated flows.



For simplicity, only plane (two-dimensional) flows will be considered. In this case, by using Cartesian coordinates

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \tag{6.72}$$

or by using cylindrical coordinates



$$v_r = \frac{\partial\phi}{\partial r} \qquad v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \tag{6.73}$$

as shown by the figure in the margin. Since we can define a stream function for plane flow, we can also let

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x} \tag{6.74}$$

or

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r} \tag{6.75}$$

where the stream function was previously defined in Eqs. 6.37 and 6.42. We know that by defining the velocities in terms of the stream function, conservation of mass is identically satisfied. If we now impose the condition of irrotationality, it follows from Eq. 6.59 that

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$$

and in terms of the stream function

$$\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)$$

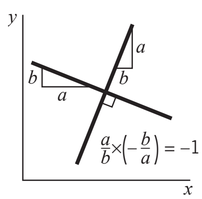
or

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Thus, for a plane irrotational flow we can use either the velocity potential or the stream function—both must satisfy Laplace's equation in two dimensions. It is apparent from these results that the velocity potential and the stream function are somehow related. We have previously shown that lines of constant ψ are streamlines; that is,

$$\left.\frac{dy}{dx}\right|_{\text{along } \psi = \text{constant}} = \frac{v}{u} \tag{6.76}$$

The change in ϕ as we move from one point (x, y) to a nearby point $(x + dx, y + dy)$ is given by the relationship

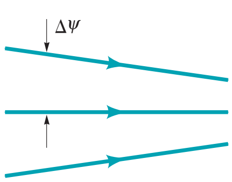


$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = u\,dx + v\,dy$$

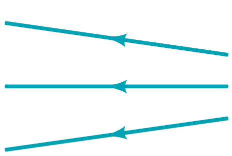
Along a line of constant ϕ we have $d\phi = 0$ so that

$$\left.\frac{dy}{dx}\right|_{\text{along } \phi = \text{constant}} = -\frac{u}{v} \tag{6.77}$$

A comparison of Eqs. 6.76 and 6.77 shows that lines of constant ϕ (called ***equipotential lines***) are orthogonal to lines of constant ψ (streamlines) at all points where they intersect. (Recall that two lines are orthogonal if the product of their slopes is -1, as illustrated by the figure in the margin.) For any potential flow field a "***flow net***" can be drawn that consists of a family of streamlines and equipotential lines. The flow net is useful in visualizing flow patterns and can be used to obtain graphical solutions by sketching in streamlines and equipotential lines and adjusting the lines until the lines are approximately orthogonal at all points where they intersect. An example of a flow net is shown in Fig. 6.15. Velocities can be estimated from the flow net, since the velocity is inversely proportional to the streamline spacing, as shown by the figure in the margin. Thus, for example, from Fig. 6.15 we can see that the velocity near the inside corner will be higher than the velocity along the outer part of the bend. (See the photographs at the beginning of Chapters 3 and 6.)



Streamwise acceleration

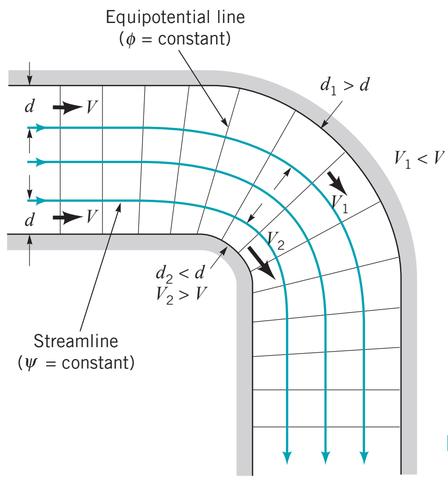


Streamwise deceleration

6.5.1 Uniform Flow

The simplest plane flow is one for which the streamlines are all straight and parallel, and the magnitude of the velocity is constant. This type of flow is called a ***uniform flow.*** For example, consider a uniform flow in the positive x direction as is illustrated in Fig. 6.16a. In this instance, $u = U$ and $v = 0$, and in terms of the velocity potential

$$\frac{\partial \phi}{\partial x} = U \qquad \frac{\partial \phi}{\partial y} = 0$$



■ **FIGURE 6.15** **Flow net for a 90° bend. (From Ref. 3, used by permission.)**

These two equations can be integrated to yield

$$\phi = Ux + C$$

where C is an arbitrary constant, which can be set equal to zero. Thus, for a uniform flow in the positive x direction

$$\phi = Ux \tag{6.78}$$

The corresponding stream function can be obtained in a similar manner, since

$$\frac{\partial \psi}{\partial y} = U \qquad \frac{\partial \psi}{\partial x} = 0$$

and, therefore,

$$\psi = Uy \tag{6.79}$$

These results can be generalized to provide the velocity potential and stream function for a uniform flow at an angle α with the x axis, as in Fig. 6.16*b*. For this case

$$\phi = U(x \cos \alpha + y \sin \alpha) \tag{6.80}$$

and

$$\psi = U(y \cos \alpha - x \sin \alpha) \tag{6.81}$$

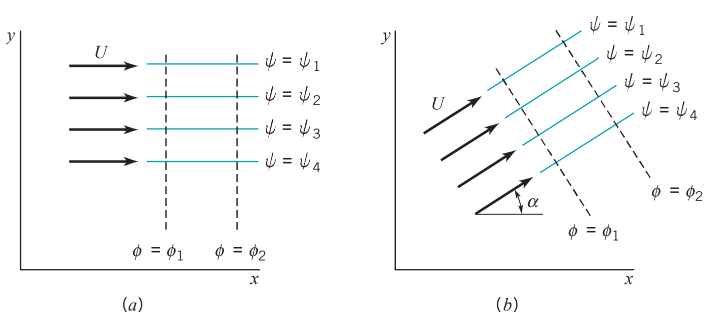
6.5.2 Source and Sink

Consider a fluid flowing radially outward from a line through the origin perpendicular to the x–y plane as is shown in Fig. 6.17. Let m be the volume rate of flow emanating from the line (per unit length), and therefore to satisfy conservation of mass

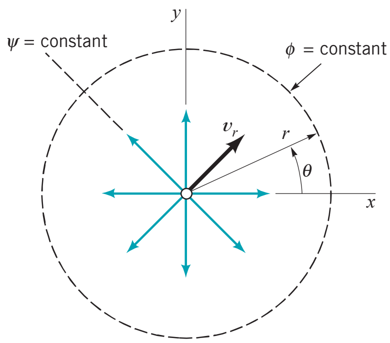
$$(2\pi r)v_r = m$$

or

$$v_r = \frac{m}{2\pi r}$$



■ **FIGURE 6.16** **Uniform flow: (*a*) in the *x* direction; (*b*) in an arbitrary direction, α.**



■ FIGURE 6.17 The streamline pattern for a source.

A source or sink represents a purely radial flow.

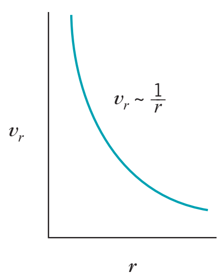
Also, since the flow is a purely radial flow, $v_\theta = 0$, the corresponding velocity potential can be obtained by integrating the equations

$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

It follows that

$$\phi = \frac{m}{2\pi} \ln r \tag{6.82}$$

If m is positive, the flow is radially outward, and the flow is considered to be a ***source*** flow. If m is negative, the flow is toward the origin, and the flow is considered to be a ***sink*** flow. The flowrate, m, is the *strength* of the source or sink.



As shown by the figure in the margin, at the origin where $r = 0$ the velocity becomes infinite, which is of course physically impossible. Thus, sources and sinks do not really exist in real flow fields, and the line representing the source or sink is a mathematical *singularity* in the flow field. However, some real flows can be approximated at points away from the origin by using sources or sinks. Also, the velocity potential representing this hypothetical flow can be combined with other basic velocity potentials to approximately describe some real flow fields. This idea is further discussed in Section 6.6.

The stream function for the source can be obtained by integrating the relationships

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \qquad v_\theta = -\frac{\partial \psi}{\partial r} = 0$$

to yield

$$\psi = \frac{m}{2\pi}\theta \tag{6.83}$$

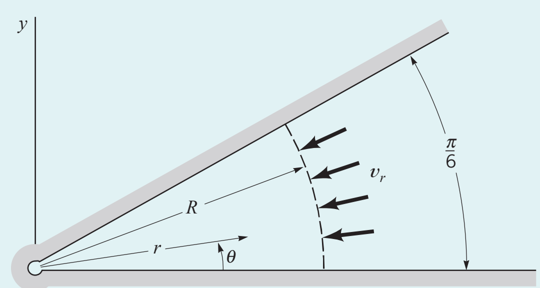
It is apparent from Eq. 6.83 that the streamlines (lines of ψ = constant) are radial lines, and from Eq. 6.82 the equipotential lines (lines of ϕ = constant) are concentric circles centered at the origin.

EXAMPLE 6.5 Potential Flow—Sink

GIVEN A nonviscous, incompressible fluid flows between wedge-shaped walls into a small opening as shown in Fig. E6.5. The velocity potential (in ft^2/s), which approximately describes this flow is

$$\phi = -2 \ln r$$

FIND Determine the volume rate of flow (per unit length) into the opening.



■ FIGURE E6.5

Solution

The components of velocity are

$$v_r = \frac{\partial \phi}{\partial r} = -\frac{2}{r} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

which indicates we have a purely radial flow. The flowrate per unit width, q, crossing the arc of length $R\pi/6$ can thus be obtained by integrating the expression

$$q = \int_0^{\pi/6} v_r R\, d\theta = -\int_0^{\pi/6} \left(\frac{2}{R}\right) R\, d\theta$$

$$= -\frac{\pi}{3} = -1.05 \text{ ft}^2/\text{s} \qquad \textbf{(Ans)}$$

COMMENT Note that the radius R is arbitrary since the flowrate crossing any curve between the two walls must be the same. The negative sign indicates that the flow is toward the opening, that is, in the negative radial direction.

6.5.3 Vortex

We next consider a flow field in which the streamlines are concentric circles—that is, we interchange the velocity potential and stream function for the source. Thus, let

$$\phi = K\theta \tag{6.84}$$

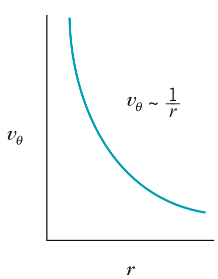
A vortex represents a flow in which the streamlines are concentric circles.

and

$$\psi = -K \ln r \tag{6.85}$$

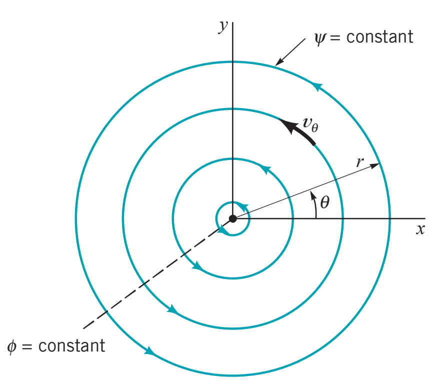
where K is a constant. In this case the streamlines are concentric circles as are illustrated in Fig. 6.18, with $v_r = 0$ and

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r} \tag{6.86}$$

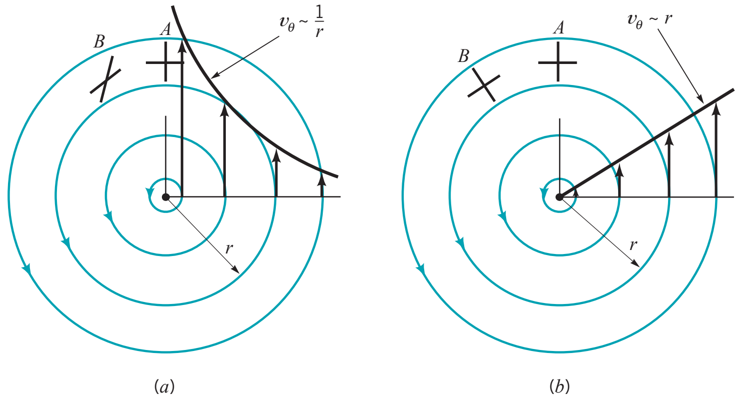


This result indicates that the tangential velocity varies inversely with the distance from the origin, as shown by the figure in the margin, with a singularity occurring at $r = 0$ (where the velocity becomes infinite).

It may seem strange that this ***vortex*** motion is irrotational (and it is since the flow field is described by a velocity potential). However, it must be recalled that rotation refers to the orientation of a fluid element and not the path followed by the element. Thus, for an irrotational vortex, if a pair of small sticks were placed in the flow field at location A, as indicated in Fig. 6.19a, the sticks would rotate as they move to location B. One of the sticks, the one that is aligned along the streamline, would follow a circular path and rotate in a counterclockwise



■ **FIGURE 6.18** **The streamline pattern for a vortex.**



■ FIGURE 6.19 Motion of fluid element from A to B: (a) for irrotational (free) vortex; (b) for rotational (forced) vortex.

direction. The other stick would rotate in a clockwise direction due to the nature of the flow field—that is, the part of the stick nearest the origin moves faster than the opposite end. Although both sticks are rotating, the average angular velocity of the two sticks is zero since the flow is irrotational.

Vortex motion can be either rotational or irrotational.

If the fluid were rotating as a rigid body, such that $v_\theta = K_1 r$ where K_1 is a constant, then sticks similarly placed in the flow field would rotate as is illustrated in Fig. 6.19b. This type of vortex motion is *rotational* and cannot be described with a velocity potential. The rotational vortex is commonly called a *forced vortex*, whereas the irrotational vortex is usually called a *free vortex*. The swirling motion of the water as it drains from a bathtub is similar to that of a free vortex, whereas the motion of a liquid contained in a tank that is rotated about its axis with angular velocity ω corresponds to a forced vortex.

A *combined vortex* is one with a forced vortex as a central core and a velocity distribution corresponding to that of a free vortex outside the core. Thus, for a combined vortex

$$v_\theta = \omega r \qquad r \leq r_0 \tag{6.87}$$

and

$$v_\theta = \frac{K}{r} \qquad r > r_0 \tag{6.88}$$

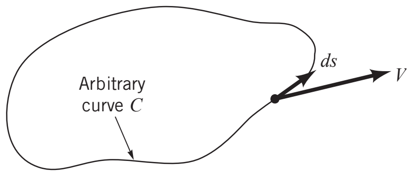
where K and ω are constants and r_0 corresponds to the radius of the central core. The pressure distribution in both the free and forced vortex was previously considered in Example 3.3. (See Fig. E6.6a for an approximation of this type of flow.)

Fluids in the News

Some hurricane facts One of the most interesting, yet potentially devastating, naturally occurring fluid flow phenomenan is a hurricane. Broadly speaking a hurricane is a rotating mass of air circulating around a low pressure central core. In some respects the motion is similar to that of a *free vortex*. The Caribbean and Gulf of Mexico experience the most hurricanes, with the official hurricane season being from June 1 to November 30. Hurricanes are usually 300 to 400 miles wide and are structured around a central eye in which the air is relatively calm. The eye is surrounded by an eye wall which is the region of strongest winds and precipitation. As one goes from the eye wall to the eye the wind speeds decrease sharply and within the eye the air is relatively calm and clear of clouds. However, in the eye the pressure is at a minimum and may be 10% less than standard atmospheric pressure. This low pressure creates strong downdrafts of dry air from above. Hurricanes are classified into five categories based on their wind speeds:

Category one—74–95 mph
Category two—96–110 mph
Category three—111–130 mph
Category four—131–155 mph
Category five—greater than 155 mph.

(See Problem 6.58.)



■ **FIGURE 6.20** **The notation for determining circulation around closed curve *C*.**

A mathematical concept commonly associated with vortex motion is that of ***circulation.*** The circulation, Γ, is defined as the line integral of the tangential component of the velocity taken around a closed curve in the flow field. In equation form, Γ can be expressed as

$$\Gamma = \oint_C \mathbf{V} \cdot d\mathbf{s} \tag{6.89}$$

where the integral sign means that the integration is taken around a closed curve, C, in the counterclockwise direction, and $d\mathbf{s}$ is a differential length along the curve as is illustrated in Fig. 6.20. For an irrotational flow, $\mathbf{V} = \nabla\phi$ so that $\mathbf{V} \cdot d\mathbf{s} = \nabla\phi \cdot d\mathbf{s} = d\phi$ and, therefore,

$$\Gamma = \oint_C d\phi = 0$$

This result indicates that for an irrotational flow the circulation will generally be zero. (Chapter 9 has further discussion of circulation in real flows.) However, if there are singularities enclosed within the curve the circulation may not be zero. For example, for the free vortex with $v_\theta = K/r$ the circulation around the circular path of radius r shown in Fig. 6.21 is

$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r\, d\theta) = 2\pi K$$

The numerical value of the circulation may depend on the particular closed path considered.

which shows that the circulation is nonzero and the constant $K = \Gamma/2\pi$. However, for irrotational flows the circulation around any path that does not include a singular point will be zero. This can be easily confirmed for the closed path $ABCD$ of Fig. 6.21 by evaluating the circulation around that path.

The velocity potential and stream function for the free vortex are commonly expressed in terms of the circulation as

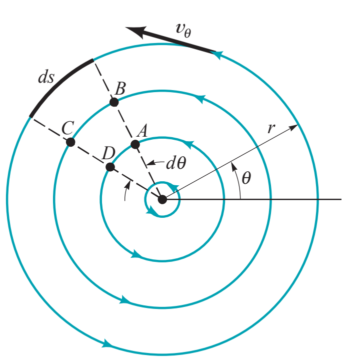
$$\phi = \frac{\Gamma}{2\pi}\theta \tag{6.90}$$

V6.4 Vortex in a beaker

and

$$\psi = -\frac{\Gamma}{2\pi}\ln r \tag{6.91}$$

The concept of circulation is often useful when evaluating the forces developed on bodies immersed in moving fluids. This application will be considered in Section 6.6.3.



■ **FIGURE 6.21** **Circulation around various paths in a free vortex.**

EXAMPLE 6.6 Potential Flow—Free Vortex

GIVEN A liquid drains from a large tank through a small opening as illustrated in Fig. E6.6*a*. A vortex forms whose velocity distribution away from the tank opening can be approximated as that of a free vortex having a velocity potential

$$\phi = \frac{\Gamma}{2\pi}\theta$$

FIND Determine an expression relating the surface shape to the strength of the vortex as specified by the circulation Γ.

SOLUTION

Since the free vortex represents an irrotational flow field, the Bernoulli equation

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

can be written between any two points. If the points are selected at the free surface, $p_1 = p_2 = 0$, so that

$$\frac{V_1^2}{2g} = z_s + \frac{V_2^2}{2g} \qquad \textbf{(1)}$$

where the free surface elevation, z_s, is measured relative to a datum passing through point (1) as shown in Fig. E6.6*b*.

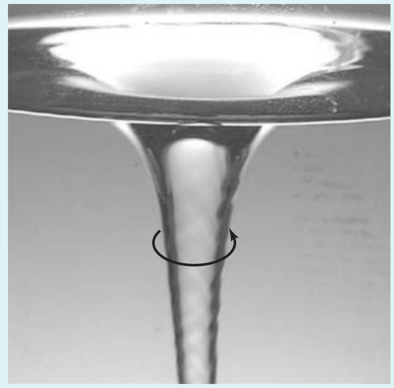
The velocity is given by the equation

$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$$

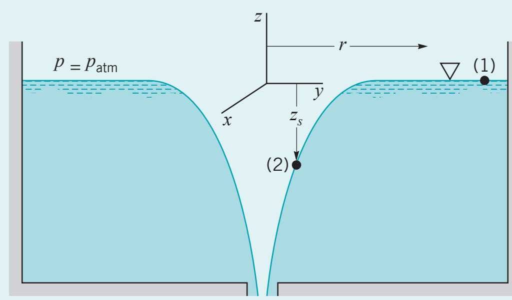
We note that far from the origin at point (1), $V_1 = v_\theta \approx 0$ so that Eq. 1 becomes

$$z_s = -\frac{\Gamma^2}{8\pi^2 r^2 g} \qquad \textbf{(Ans)}$$

which is the desired equation for the surface profile.



■ FIGURE E6.6*a*



■ FIGURE E6.6*b*

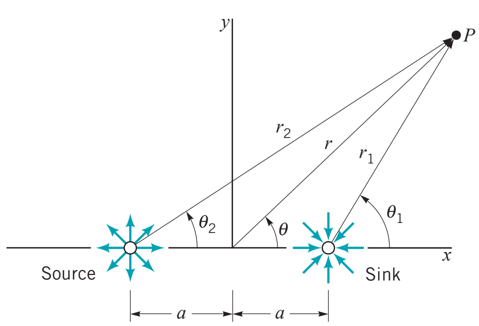
COMMENT The negative sign indicates that the surface falls as the origin is approached as shown in Fig. E6.6. This solution is not valid very near the origin since the predicted velocity becomes excessively large as the origin is approached.

6.5.4 Doublet

A doublet is formed by an appropriate source–sink pair.

The final, basic potential flow to be considered is one that is formed by combining a source and sink in a special way. Consider the equal strength, source–sink pair of Fig. 6.22. The combined stream function for the pair is

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$



■ **FIGURE 6.22 The combination of a source and sink of equal strength located along the *x* axis.**

which can be rewritten as

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} \tag{6.92}$$

From Fig. 6.22 it follows that

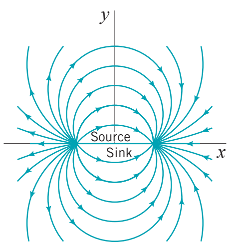
$$\tan\theta_1 = \frac{r\sin\theta}{r\cos\theta - a}$$

and

$$\tan\theta_2 = \frac{r\sin\theta}{r\cos\theta + a}$$

These results substituted into Eq. 6.92 give

$$\tan\left(-\frac{2\pi\psi}{m}\right) = \frac{2ar\sin\theta}{r^2 - a^2}$$



so that

$$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right) \tag{6.93}$$

The figure in the margin shows typical streamlines for this flow. For small values of the distance a

$$\psi = -\frac{m}{2\pi}\frac{2ar\sin\theta}{r^2 - a^2} = -\frac{mar\sin\theta}{\pi(r^2 - a^2)} \tag{6.94}$$

since the tangent of an angle approaches the value of the angle for small angles.

A doublet is formed by letting a source and sink approach one another.

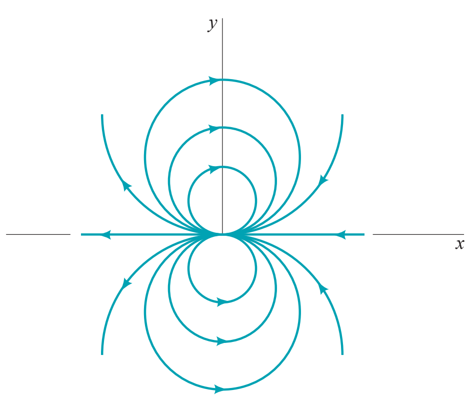
The so-called ***doublet*** is formed by letting the source and sink approach one another ($a \to 0$) while increasing the strength m ($m \to \infty$) so that the product ma/π remains constant. In this case, since $r/(r^2 - a^2) \to 1/r$, Eq. 6.94 reduces to

$$\psi = -\frac{K\sin\theta}{r} \tag{6.95}$$

where K, a constant equal to ma/π, is called the *strength* of the doublet. The corresponding velocity potential for the doublet is

$$\phi = \frac{K\cos\theta}{r} \tag{6.96}$$

Plots of lines of constant ψ reveal that the streamlines for a doublet are circles through the origin tangent to the x axis as shown in Fig. 6.23. Just as sources and sinks are not physically realistic entities, neither are doublets. However, the doublet when combined with other basic potential flows



■ FIGURE 6.23 Streamlines for a doublet.

■ **TABLE 6.1**
Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see Fig. 6.16*b*)	$\phi = U(x\cos\alpha + y\sin\alpha)$	$\psi = U(y\cos\alpha - x\sin\alpha)$	$u = U\cos\alpha$ $v = U\sin\alpha$
Source or sink (see Fig. 6.17) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi}\ln r$	$\psi = \dfrac{m}{2\pi}\theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see Fig. 6.18) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi}\theta$	$\psi = -\dfrac{\Gamma}{2\pi}\ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see Fig. 6.23)	$\phi = \dfrac{K\cos\theta}{r}$	$\psi = -\dfrac{K\sin\theta}{r}$	$v_r = -\dfrac{K\cos\theta}{r^2}$ $v_\theta = -\dfrac{K\sin\theta}{r^2}$

[a]Velocity components are related to the velocity potential and stream function through the relationships:
$u = \dfrac{\partial\phi}{\partial x} = \dfrac{\partial\psi}{\partial y} \qquad v = \dfrac{\partial\phi}{\partial y} = -\dfrac{\partial\psi}{\partial x} \qquad v_r = \dfrac{\partial\phi}{\partial r} = \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \qquad v_\theta = \dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta} = -\dfrac{\partial\psi}{\partial r}.$

provides a useful representation of some flow fields of practical interest. For example, we will determine in Section 6.6.3 that the combination of a uniform flow and a doublet can be used to represent the flow around a circular cylinder. Table 6.1 provides a summary of the pertinent equations for the basic, plane potential flows considered in the preceding sections.

6.6 Superposition of Basic, Plane Potential Flows

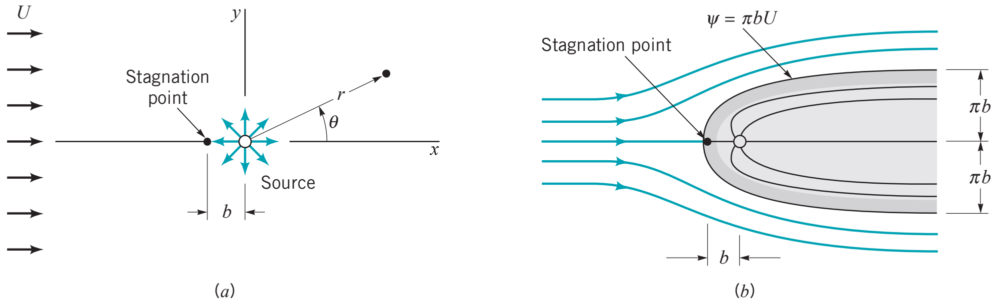
As was discussed in the previous section, potential flows are governed by Laplace's equation, which is a linear partial differential equation. It therefore follows that the various basic velocity potentials and stream functions can be combined to form new potentials and stream functions. (Why is this true?) Whether such combinations yield useful results remains to be seen. It is to be noted that *any streamline in an inviscid flow field can be considered as a solid boundary*, since the conditions along a solid boundary and a streamline are the same—that is, there is no flow through the boundary or the streamline. Thus, if we can combine some of the basic velocity potentials or stream functions to yield a streamline that corresponds to a particular body shape of interest, that combination can be used to describe in detail the flow around that body. This method of solving some interesting flow problems, commonly called the ***method of superposition,*** is illustrated in the following three sections.

6.6.1 Source in a Uniform Stream—Half-Body

Flow around a half-body is obtained by the addition of a source to a uniform flow.

Consider the superposition of a source and a uniform flow as shown in Fig. 6.24*a*. The resulting stream function is

$$\psi = \psi_{\text{uniform flow}} + \psi_{\text{source}}$$
$$= Ur\sin\theta + \frac{m}{2\pi}\theta \tag{6.97}$$



■ **FIGURE 6.24 The flow around a half-body: (*a*) superposition of a source and a uniform flow; (*b*) replacement of streamline $\psi = \pi bU$ with solid boundary to form half-body.**

and the corresponding velocity potential is

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r \tag{6.98}$$

V6.5 Half-body

It is clear that at some point along the negative x axis the velocity due to the source will just cancel that due to the uniform flow and a stagnation point will be created. For the source alone

$$v_r = \frac{m}{2\pi r}$$

so that the stagnation point will occur at $x = -b$ where

$$U = \frac{m}{2\pi b}$$

or

$$b = \frac{m}{2\pi U} \tag{6.99}$$

The value of the stream function at the stagnation point can be obtained by evaluating ψ at $r = b$ and $\theta = \pi$, which yields from Eq. 6.97

$$\psi_{\text{stagnation}} = \frac{m}{2}$$

Since $m/2 = \pi bU$ (from Eq. 6.99) it follows that the equation of the streamline passing through the stagnation point is

$$\pi bU = Ur\sin\theta + bU\theta$$

or

$$r = \frac{b(\pi - \theta)}{\sin\theta} \tag{6.100}$$

For inviscid flow, a streamline can be replaced by a solid boundary.

where θ can vary between 0 and 2π. A plot of this streamline is shown in Fig. 6.24*b*. If we replace this streamline with a solid boundary, as indicated in the figure, then it is clear that this combination of a uniform flow and a source can be used to describe the flow around a streamlined body placed in a uniform stream. The body is open at the downstream end, and thus is called a ***half-body.*** Other streamlines in the flow field can be obtained by setting ψ = constant in Eq. 6.97 and plotting the resulting equation. A number of these streamlines are shown in Fig. 6.24*b*. Although the streamlines inside the body are shown, they are actually of no interest in this case, since we are concerned with the flow field outside the body. It should be noted that the singularity in the flow field (the source) occurs inside the body, and there are no singularities in the flow field of interest (outside the body).

The width of the half-body asymptotically approaches $2\pi b$. This follows from Eq. 6.100, which can be written as

$$y = b(\pi - \theta)$$

so that as $\theta \to 0$ or $\theta \to 2\pi$ the half-width approaches $\pm b\pi$. With the stream function (or velocity potential) known, the velocity components at any point can be obtained. For the half-body, using the stream function given by Eq. 6.97,

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = U \cos \theta + \frac{m}{2\pi r}$$

and

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U \sin \theta$$

Thus, the square of the magnitude of the velocity, V, at any point is

$$V^2 = v_r^2 + v_\theta^2 = U^2 + \frac{Um \cos \theta}{\pi r} + \left(\frac{m}{2\pi r}\right)^2$$

and since $b = m/2\pi U$

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos \theta + \frac{b^2}{r^2}\right) \tag{6.101}$$

With the velocity known, the pressure at any point can be determined from the Bernoulli equation, which can be written between any two points in the flow field since the flow is irrotational. Thus, applying the Bernoulli equation between a point far from the body, where the pressure is p_0 and the velocity is U, and some arbitrary point with pressure p and velocity V, it follows that

$$p_0 + \tfrac{1}{2}\rho U^2 = p + \tfrac{1}{2}\rho V^2 \tag{6.102}$$

where elevation changes have been neglected. Equation 6.101 can now be substituted into Eq. 6.102 to obtain the pressure at any point in terms of the reference pressure, p_0, and the upstream velocity, U.

For a potential flow the fluid is allowed to slip past a fixed solid boundary.

This relatively simple potential flow provides some useful information about the flow around the front part of a streamlined body, such as a bridge pier or strut placed in a uniform stream. An important point to be noted is that the velocity tangent to the surface of the body is not zero; that is, the fluid "slips" by the boundary. This result is a consequence of neglecting viscosity, the fluid property that causes real fluids to stick to the boundary, thus creating a "no-slip" condition. All potential flows differ from the flow of real fluids in this respect and do not accurately represent the velocity very near the boundary. However, outside this very thin boundary layer the velocity distribution will generally correspond to that predicted by potential flow theory if flow separation does not occur. (See Section 9.2.6.) Also, the pressure distribution along the surface will closely approximate that predicted from the potential flow theory, since the boundary layer is thin and there is little opportunity for the pressure to vary through the thin layer. In fact, as discussed in more detail in Chapter 9, the pressure distribution obtained from potential flow theory is used in conjunction with viscous flow theory to determine the nature of flow within the boundary layer.

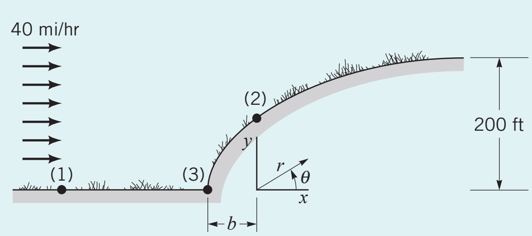
EXAMPLE 6.7 Potential Flow—Half-body

GIVEN A 40 mi/hr wind blows toward a hill arising from a plain that can be approximated with the top section of a half-body as illustrated in Fig. E6.7*a*. The height of the hill approaches 200 ft as shown. Assume an air density of 0.00238 slugs/ft^3.

FIND

(a) What is the magnitude of the air velocity at a point on the hill directly above the origin [point (2)]?

(b) What is the elevation of point (2) above the plain and what is the difference in pressure between point (1) on the plain far from the hill and point (2)?



■ FIGURE E6.7*a*

Solution

(a) The velocity is given by Eq. 6.101 as

$$V^2 = U^2\left(1 + 2\frac{b}{r}\cos\theta + \frac{b^2}{r^2}\right)$$

At point (2), $\theta = \pi/2$, and since this point is on the surface (Eq. 6.100)

$$r = \frac{b(\pi - \theta)}{\sin\theta} = \frac{\pi b}{2} \quad \textbf{(1)}$$

Thus,

$$V_2^2 = U^2\left[1 + \frac{b^2}{(\pi b/2)^2}\right] = U^2\left(1 + \frac{4}{\pi^2}\right)$$

and the magnitude of the velocity at (2) for a 40 mi/hr approaching wind is

$$V_2 = \left(1 + \frac{4}{\pi^2}\right)^{1/2}(40 \text{ mi/hr}) = 47.4 \text{ mi/hr} \quad \textbf{(Ans)}$$

(b) The elevation at (2) above the plain is given by Eq. 1 as

$$y_2 = \frac{\pi b}{2}$$

Since the height of the hill approaches 200 ft and this height is equal to πb, it follows that

$$y_2 = \frac{200 \text{ ft}}{2} = 100 \text{ ft} \quad \textbf{(Ans)}$$

From the Bernoulli equation (with the y axis the vertical axis)

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + y_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + y_2$$

so that

$$p_1 - p_2 = \frac{\rho}{2}(V_2^2 - V_1^2) + \gamma(y_2 - y_1)$$

and with

$$V_1 = (40 \text{ mi/hr})\left(\frac{5280 \text{ ft/mi}}{3600 \text{ s/hr}}\right) = 58.7 \text{ ft/s}$$

and

$$V_2 = (47.4 \text{ mi/hr})\left(\frac{5280 \text{ ft/mi}}{3600 \text{ s/hr}}\right) = 69.5 \text{ ft/s}$$

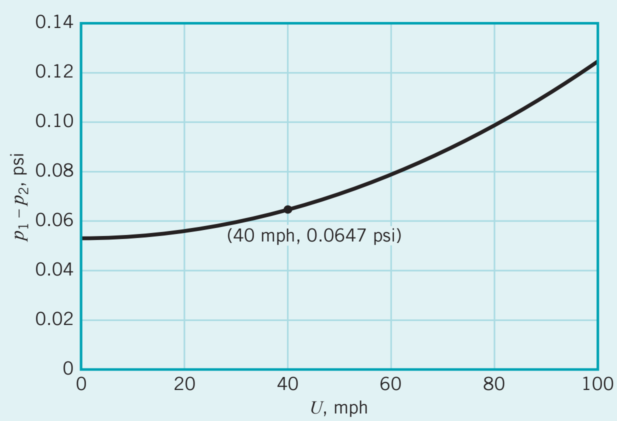
it follows that

$$\begin{aligned} p_1 - p_2 &= \frac{(0.00238 \text{ slugs/ft}^3)}{2}[(69.5 \text{ ft/s})^2 - (58.7 \text{ ft/s})^2] \\ &\quad + (0.00238 \text{ slugs/ft}^3)(32.2 \text{ ft/s}^2)(100 \text{ ft} - 0 \text{ ft}) \\ &= 9.31 \text{ lb/ft}^2 = 0.0647 \text{ psi} \end{aligned} \quad \textbf{(Ans)}$$

COMMENTS This result indicates that the pressure on the hill at point (2) is slightly lower than the pressure on the plain at some distance from the base of the hill with a 0.0533 psi difference due to the elevation increase and a 0.0114 psi difference due to the velocity increase.

By repeating the calculations for various values of the upstream wind speed, U, the results shown in Fig. E6.7*b* are obtained. Note that as the wind speed increases, the pressure difference increases from the calm conditions of $p_1 - p_2 = 0.0533$ psi.

The maximum velocity along the hill surface does not occur at point (2) but farther up the hill at $\theta = 63°$. At this point $V_{\text{surface}} = 1.26U$. The minimum velocity ($V = 0$) and maximum pressure occur at point (3), the stagnation point.



■ **FIGURE E6.7*b***

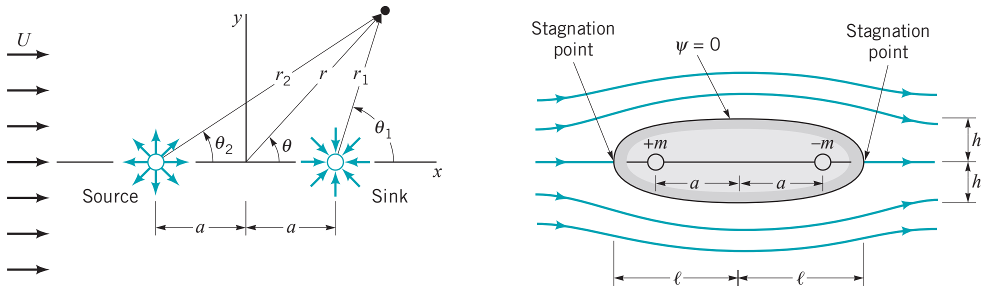
6.6.2 Rankine Ovals

The half-body described in the previous section is a body that is "open" at one end. To study the flow around a closed body, a source and a sink of equal strength can be combined with a uniform flow as shown in Fig. 6.25*a*. The stream function for this combination is

$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2) \quad \textbf{(6.103)}$$

and the velocity potential is

$$\phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2) \quad \textbf{(6.104)}$$



(*a*) (*b*)

■ **FIGURE 6.25** **The flow around a Rankine oval: (*a*) superposition of source–sink pair and a uniform flow; (*b*) replacement of streamline $\psi = 0$ with solid boundary to form Rankine oval.**

As discussed in Section 6.5.4, the stream function for the source–sink pair can be expressed as in Eq. 6.93 and, therefore, Eq. 6.103 can also be written as

$$\psi = Ur\sin\theta - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

or

$$\psi = Uy - \frac{m}{2\pi}\tan^{-1}\left(\frac{2ay}{x^2 + y^2 - a^2}\right) \qquad \textbf{(6.105)}$$

Rankine ovals are formed by combining a source and sink with a uniform flow.

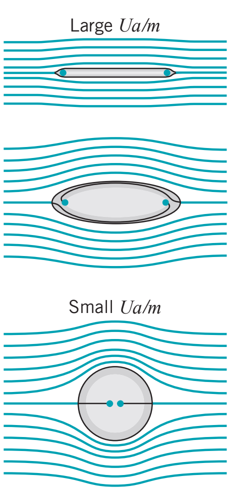
The corresponding streamlines for this flow field are obtained by setting ψ = constant. If several of these streamlines are plotted, it will be discovered that the streamline $\psi = 0$ forms a closed body as is illustrated in Fig. 6.25*b*. We can think of this streamline as forming the surface of a body of length 2ℓ and width $2h$ placed in a uniform stream. The streamlines inside the body are of no practical interest and are not shown. Note that since the body is closed, all of the flow emanating from the source flows into the sink. These bodies have an oval shape and are termed ***Rankine ovals.***

Stagnation points occur at the upstream and downstream ends of the body as are indicated in Fig. 6.25*b*. These points can be located by determining where along the *x* axis the velocity is zero. The stagnation points correspond to the points where the uniform velocity, the source velocity, and the sink velocity all combine to give a zero velocity. The locations of the stagnation points depend on the value of *a*, *m*, and *U*. The body half-length, ℓ (the value of $|x|$ that gives $\mathbf{V} = 0$ when $y = 0$), can be expressed as

$$\ell = \left(\frac{ma}{\pi U} + a^2\right)^{1/2} \qquad \textbf{(6.106)}$$

or

$$\frac{\ell}{a} = \left(\frac{m}{\pi U a} + 1\right)^{1/2} \qquad \textbf{(6.107)}$$



The body half-width, *h*, can be obtained by determining the value of *y* where the *y* axis intersects the $\psi = 0$ streamline. Thus, from Eq. 6.105 with $\psi = 0$, $x = 0$, and $y = h$, it follows that

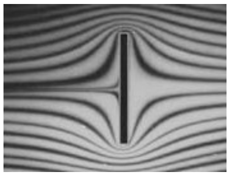
$$h = \frac{h^2 - a^2}{2a}\tan\frac{2\pi U h}{m} \qquad \textbf{(6.108)}$$

or

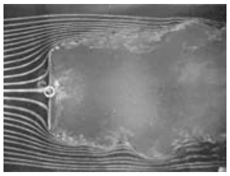
$$\frac{h}{a} = \frac{1}{2}\left[\left(\frac{h}{a}\right)^2 - 1\right]\tan\left[2\left(\frac{\pi U a}{m}\right)\frac{h}{a}\right] \qquad \textbf{(6.109)}$$

Equations 6.107 and 6.109 show that both ℓ/a and h/a are functions of the dimensionless parameter, $\pi Ua/m$. Although for a given value of Ua/m the corresponding value of ℓ/a can be determined directly from Eq. 6.107, h/a must be determined by a trial and error solution of Eq. 6.109.

A large variety of body shapes with different length to width ratios can be obtained by using different values of Ua/m, as shown by the figure in the margin. As this parameter becomes large, flow



Potential Flow



Viscous Flow

around a long slender body is described, whereas for small values of the parameter, flow around a more blunt shape is obtained. Downstream from the point of maximum body width the surface pressure increases with distance along the surface. This condition (called an adverse pressure gradient) typically leads to separation of the flow from the surface, resulting in a large low pressure wake on the downstream side of the body. Separation is not predicted by potential theory (which simply indicates a symmetrical flow). This is illustrated by the figure in the margin for an extreme blunt shape. Therefore, the potential solution for the Rankine ovals will give a reasonable approximation of the velocity outside the thin, viscous boundary layer and the pressure distribution on the front part of the body only.

6.6.3 Flow around a Circular Cylinder

A doublet combined with a uniform flow can be used to represent flow around a circular cylinder.

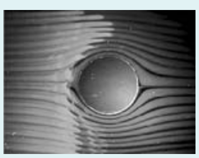
As was noted in the previous section, when the distance between the source–sink pair approaches zero, the shape of the Rankine oval becomes more blunt and in fact approaches a circular shape. Since the doublet described in Section 6.5.4 was developed by letting a source–sink pair approach one another, it might be expected that a uniform flow in the positive x direction combined with a doublet could be used to represent flow around a circular cylinder. This combination gives for the stream function

$$\psi = Ur \sin\theta - \frac{K \sin\theta}{r} \tag{6.110}$$

and for the velocity potential

$$\phi = Ur \cos\theta + \frac{K \cos\theta}{r} \tag{6.111}$$

V6.6 Circular cylinder



In order for the stream function to represent flow around a circular cylinder it is necessary that ψ = constant for $r = a$, where a is the radius of the cylinder. Since Eq. 6.110 can be written as

$$\psi = \left(U - \frac{K}{r^2}\right) r \sin\theta$$

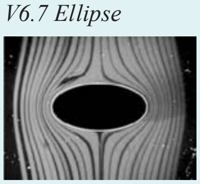
it follows that $\psi = 0$ for $r = a$ if

$$U - \frac{K}{a^2} = 0$$

which indicates that the doublet strength, K, must be equal to Ua^2. Thus, the stream function for flow around a circular cylinder can be expressed as

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta \tag{6.112}$$

V6.7 Ellipse



and the corresponding velocity potential is

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta \tag{6.113}$$

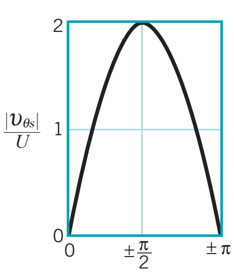
A sketch of the streamlines for this flow field is shown in Fig. 6.26.

The velocity components can be obtained from either Eq. 6.112 or 6.113 as

$$v_r = \frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \tag{6.114}$$

and

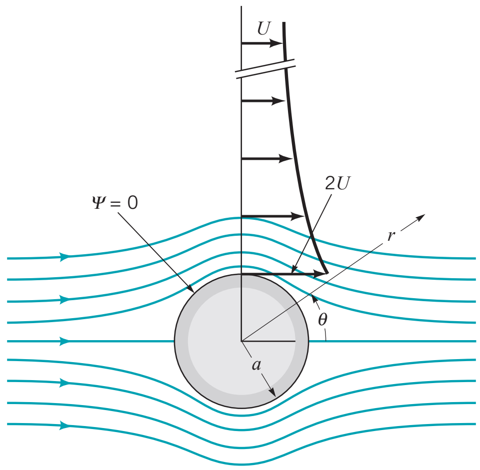
$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta \tag{6.115}$$



On the surface of the cylinder ($r = a$) it follows from Eq. 6.114 and 6.115 that $v_r = 0$ and

$$v_{\theta s} = -2U \sin\theta$$

As shown by the figure in the margin, the maximum velocity occurs at the top and bottom of the cylinder ($\theta = \pm\pi/2$) and has a magnitude of twice the upstream velocity, U. As we move away from the cylinder along the ray $\theta = \pi/2$ the velocity varies, as is illustrated in Fig. 6.26.



■ FIGURE 6.26 The flow around a circular cylinder.

The pressure distribution on the cylinder surface is obtained from the Bernoulli equation.

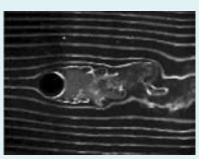
The pressure distribution on the cylinder surface is obtained from the Bernoulli equation written from a point far from the cylinder where the pressure is p_0 and the velocity is U so that

$$p_0 + \tfrac{1}{2}\rho U^2 = p_s + \tfrac{1}{2}\rho v_{\theta s}^2$$

where p_s is the surface pressure. Elevation changes are neglected. Since $v_{\theta s} = -2U \sin\theta$, the surface pressure can be expressed as

$$p_s = p_0 + \tfrac{1}{2}\rho U^2(1 - 4\sin^2\theta) \tag{6.116}$$

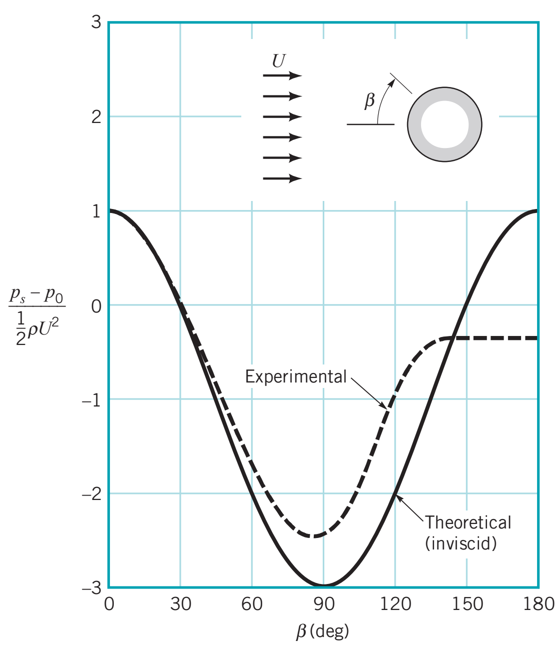
V6.8 Circular cylinder with separation



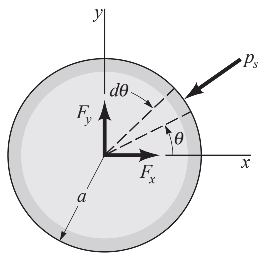
A comparison of this theoretical, symmetrical pressure distribution expressed in dimensionless form with a typical measured distribution is shown in Fig. 6.27. This figure clearly reveals that only on the upstream part of the cylinder is there approximate agreement between the potential flow and the experimental results. Because of the viscous boundary layer that develops on the cylinder, the main flow separates from the surface of the cylinder, leading to the large difference between the theoretical, frictionless fluid solution and the experimental results on the downstream side of the cylinder (see Chapter 9).

The resultant force (per unit length) developed on the cylinder can be determined by integrating the pressure over the surface. From Fig. 6.28 it can be seen that

$$F_x = -\int_0^{2\pi} p_s \cos\theta \, a \, d\theta \tag{6.117}$$



■ FIGURE 6.27 A comparison of theoretical (inviscid) pressure distribution on the surface of a circular cylinder with typical experimental distribution.



■ **FIGURE 6.28** **The notation for determining lift and drag on a circular cylinder.**

V6.9 Potential and viscous flow

and

$$F_y = -\int_0^{2\pi} p_s \sin\theta \, a \, d\theta \qquad \textbf{(6.118)}$$

Potential theory incorrectly predicts that the drag on a cylinder is zero.

where F_x is the *drag* (force parallel to direction of the uniform flow) and F_y is the *lift* (force perpendicular to the direction of the uniform flow). Substitution for p_s from Eq. 6.116 into these two equations, and subsequent integration, reveals that $F_x = 0$ and $F_y = 0$ (Problem 6.73). These results indicate that both the drag and lift as predicted by potential theory for a fixed cylinder in a uniform stream are zero. Since the pressure distribution is symmetrical around the cylinder, this is not really a surprising result. However, we know from experience that there is a significant drag developed on a cylinder when it is placed in a moving fluid. This discrepancy is known as d'Alembert's paradox. The paradox is named after Jean le Rond d'Alembert (1717–1783), a French mathematician and philosopher, who first showed that the drag on bodies immersed in inviscid fluids is zero. It was not until the latter part of the nineteenth century and the early part of the twentieth century that the role viscosity plays in the steady fluid motion was understood and d'Alembert's paradox explained (see Section 9.1).

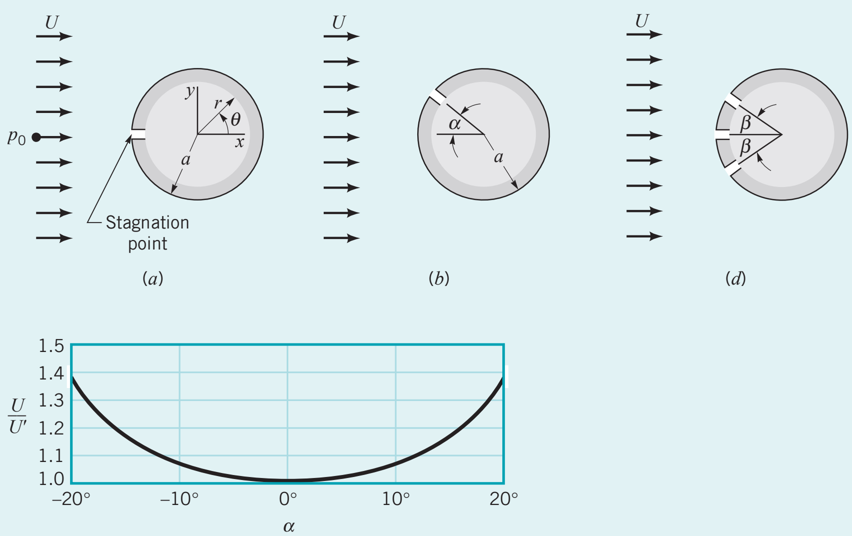
EXAMPLE 6.8 Potential Flow—Cylinder

GIVEN When a circular cylinder is placed in a uniform stream, a stagnation point is created on the cylinder as is shown in Fig. E6.8*a*. If a small hole is located at this point, the stagnation pressure, p_{stag}, can be measured and used to determine the approach velocity, U.

FIND

(a) Show how p_{stag} and U are related.

(b) If the cylinder is misaligned by an angle α (Figure E6.8*b*), but the measured pressure is still interpreted as the stagnation pressure, determine an expression for the ratio of the true velocity, U, to the predicted velocity, U'. Plot this ratio as a function of α for the range $-20° \le \alpha \le 20°$.



■ **FIGURE E6.8**

SOLUTION

(a) The velocity at the stagnation point is zero so the Bernoulli equation written between a point on the stagnation streamline upstream from the cylinder and the stagnation point gives

$$\frac{p_0}{\gamma} + \frac{U^2}{2g} = \frac{p_{\text{stag}}}{\gamma}$$

Thus,

$$U = \left[\frac{2}{\rho}(p_{\text{stag}} - p_0)\right]^{1/2} \qquad \textbf{(Ans)}$$

COMMENT A measurement of the difference between the pressure at the stagnation point and the upstream pressure can be used to measure the approach velocity. This is, of course, the same result that was obtained in Section 3.5 for Pitot-static tubes.

(b) If the direction of the fluid approaching the cylinder is not known precisely, it is possible that the cylinder is misaligned by some angle, α. In this instance the pressure actually measured, p_α, will be different from the stagnation pressure, but if the misalignment is not recognized the predicted approach velocity, U', would still be calculated as

$$U' = \left[\frac{2}{\rho}(p_\alpha - p_0)\right]^{1/2}$$

Thus,

$$\frac{U(\text{true})}{U'(\text{predicted})} = \left(\frac{p_{\text{stag}} - p_0}{p_\alpha - p_0}\right)^{1/2} \qquad \textbf{(1)}$$

The velocity on the surface of the cylinder, v_θ, where $r = a$, is obtained from Eq. 6.115 as

$$v_\theta = -2U \sin\theta$$

If we now write the Bernoulli equation between a point upstream of the cylinder and the point on the cylinder where $r = a, \theta = \alpha$, it follows that

$$p_0 + \frac{1}{2}\rho U^2 = p_\alpha + \frac{1}{2}\rho(-2U\sin\alpha)^2$$

and, therefore,

$$p_\alpha - p_0 = \tfrac{1}{2}\rho U^2(1 - 4\sin^2\alpha) \qquad \textbf{(2)}$$

Since $p_{\text{stag}} - p_0 = \frac{1}{2}\rho U^2$ it follows from Eqs. 1 and 2 that

$$\frac{U(\text{true})}{U'(\text{predicted})} = (1 - 4\sin^2\alpha)^{-1/2} \qquad \textbf{(Ans)}$$

This velocity ratio is plotted as a function of the misalignment angle α in Fig. E6.8*c*.

COMMENT It is clear from these results that significant errors can arise if the stagnation pressure tap is not aligned with the stagnation streamline. As is discussed in Section 3.5, if two additional, symmetrically located holes are drilled on the cylinder, as are illustrated in Fig. E6.8*d*, the correct orientation of the cylinder can be determined. The cylinder is rotated until the pressures in the two symmetrically placed holes are equal, thus indicating that the center hole coincides with the stagnation streamline. For $\beta = 30°$ the pressure at the two holes theoretically corresponds to the upstream pressure, p_0. With this orientation a measurement of the difference in pressure between the center hole and the side holes can be used to determine U.

An additional, interesting potential flow can be developed by adding a free vortex to the stream function or velocity potential for the flow around a cylinder. In this case

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta - \frac{\Gamma}{2\pi}\ln r \qquad \textbf{(6.119)}$$

and

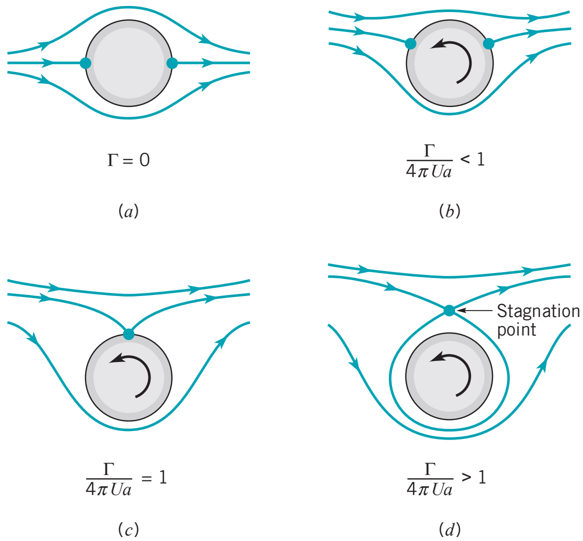
$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta \qquad \textbf{(6.120)}$$

where Γ is the circulation. We note that the circle $r = a$ will still be a streamline (and thus can be replaced with a solid cylinder), since the streamlines for the added free vortex are all circular. However, the tangential velocity, v_θ, on the surface of the cylinder $(r = a)$ now becomes

$$v_{\theta s} = -\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = -2U\sin\theta + \frac{\Gamma}{2\pi a} \qquad \textbf{(6.121)}$$

Flow around a rotating cylinder is approximated by the addition of a free vortex.

This type of flow field could be approximately created by placing a rotating cylinder in a uniform stream. Because of the presence of viscosity in any real fluid, the fluid in contact with the rotating cylinder would rotate with the same velocity as the cylinder, and the resulting flow field would resemble that developed by the combination of a uniform flow past a cylinder and a free vortex.



■ **FIGURE 6.29** **The location of stagnation points on a circular cylinder: (*a*) without circulation; (*b*, *c*, *d*) with circulation.**

A variety of streamline patterns can be developed, depending on the vortex strength, Γ. For example, from Eq. 6.121 we can determine the location of stagnation points on the surface of the cylinder. These points will occur at $\theta = \theta_{\text{stag}}$ where $v_\theta = 0$ and therefore from Eq. 6.121

$$\sin \theta_{\text{stag}} = \frac{\Gamma}{4\pi U a} \tag{6.122}$$

If $\Gamma = 0$, then $\theta_{\text{stag}} = 0$ or π—that is, the stagnation points occur at the front and rear of the cylinder as are shown in Fig. 6.29*a*. However, for $-1 \leq \Gamma/4\pi U a \leq 1$, the stagnation points will occur at some other location on the surface as illustrated in Figs. 6.29*b*,*c*. If the absolute value of the parameter $\Gamma/4\pi U a$ exceeds 1, Eq. 6.122 cannot be satisfied, and the stagnation point is located away from the cylinder as shown in Fig. 6.29*d*.

The force per unit length developed on the cylinder can again be obtained by integrating the differential pressure forces around the circumference as in Eqs. 6.117 and 6.118. For the cylinder with circulation, the surface pressure, p_s, is obtained from the Bernoulli equation (with the surface velocity given by Eq. 6.121)

$$p_0 + \frac{1}{2}\rho U^2 = p_s + \frac{1}{2}\rho\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

or

$$p_s = p_0 + \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta + \frac{2\Gamma\sin\theta}{\pi a U} - \frac{\Gamma^2}{4\pi^2 a^2 U^2}\right) \tag{6.123}$$

Equation 6.123 substituted into Eq. 6.117 for the drag, and integrated, again yields (Problem 6.74)

$$F_x = 0$$

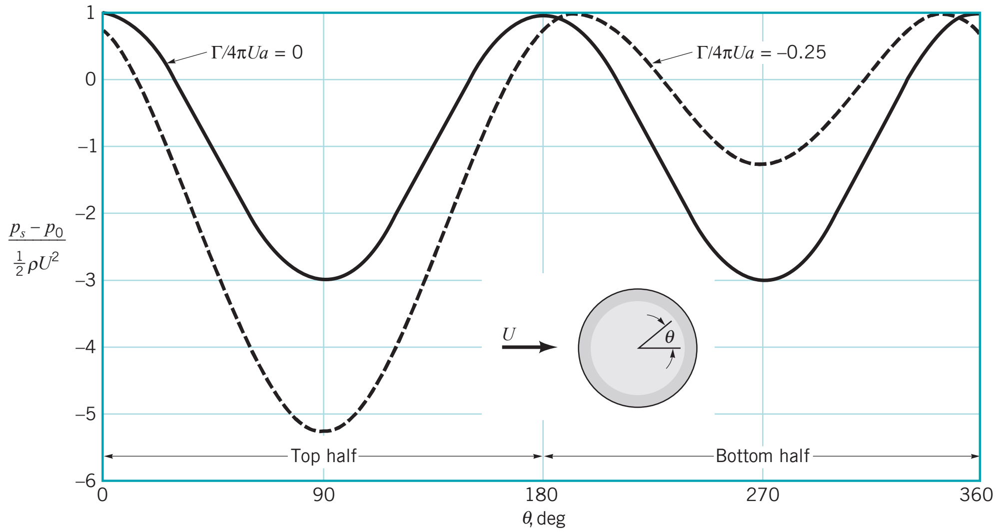
Potential flow past a cylinder with circulation gives zero drag but non-zero lift.

That is, even for the rotating cylinder no force in the direction of the uniform flow is developed. However, use of Eq. 6.123 with the equation for the lift, F_y (Eq. 6.118), yields (Problem 6.74)

$$F_y = -\rho U \Gamma \tag{6.124}$$

Thus, for the cylinder with circulation, lift is developed equal to the product of the fluid density, the upstream velocity, and the circulation. The negative sign means that if U is positive (in the positive x direction) and Γ is positive (a free vortex with counterclockwise rotation), the direction of the F_y is downward.

Of course, if the cylinder is rotated in the clockwise direction ($\Gamma < 0$) the direction of F_y would be upward. This can be seen by studying the surface pressure distribution (Eq. 6.123), which is plotted in Fig. 6.30 for two situations. One has $\Gamma/4\pi U a = 0$, which corresponds to no rotation of the cylinder. The other has $\Gamma/4\pi U a = -0.25$, which corresponds to clockwise rotation of the cylinder. With no



■ **FIGURE 6.30** **Pressure distribution on a circular cylinder with and without rotation.**

rotation the flow is symmetrical both top to bottom and front to back on the cylinder. With rotation the flow is symmetrical front to back, but not top to bottom. In this case the two stagnation points [i.e., $(p_s - p_0)/(\rho U^2/2) = 1$] are located on the bottom of the cylinder and the average pressure on the top half of the cylinder is less than that on the bottom half. The result is an upward lift force. It is this force acting in a direction perpendicular to the direction of the approach velocity that causes baseballs and golf balls to curve when they spin as they are propelled through the air. The development of this lift on rotating bodies is called the *Magnus effect.* (See Section 9.4 for further comments.)

Although Eq. 6.124 was developed for a cylinder with circulation, it gives the lift per unit length for any two-dimensional object of any cross-sectional shape placed in a uniform, inviscid stream. The circulation is determined around any closed curve containing the body. The generalized equation relating lift to fluid density, velocity, and circulation is called the *Kutta–Joukowski law,* and is commonly used to determine the lift on airfoils (see Section 9.4.2 and Refs. 2–6).

Fluids in the News

A sailing ship without sails A sphere or cylinder spinning about its axis when placed in an airstream develops a force at right angles to the direction of the airstream. This phenomenon is commonly referred to as the *Magnus effect* and is responsible for the curved paths of baseballs and golf balls. Another lesser-known application of the Magnus effect was proposed by a German physicist and engineer, Anton Flettner, in the 1920s. Flettner's idea was to use the Magnus effect to make a ship move. To demonstrate the practicality of the "rotor-ship" he purchased a sailing schooner and replaced the ship's masts and rigging with two vertical cylinders that were 50 feet high and 9 feet in diameter. The cylinders looked like smokestacks on the ship. Their spinning motion was developed by 45-hp motors. The combination of a wind and the rotating cylinders created a force (Magnus effect) to push the ship forward. The ship, named the *Baden Baden*, made a successful voyage across the Atlantic, arriving in New York Harbor on May 9, 1926. Although the feasibility of the rotor-ship was clearly demonstrated, it proved to be less efficient and practical than more conventional vessels and the idea was not pursued. (See Problem 6.72.)

6.7 Other Aspects of Potential Flow Analysis

In the preceding section the method of superposition of basic potentials has been used to obtain detailed descriptions of irrotational flow around certain body shapes immersed in a uniform stream. For the cases considered, two or more of the basic potentials were combined and the question is asked: What kind of flow does this combination represent? This approach is relatively simple and does not require the use of advanced mathematical techniques. It is, however, restrictive in its general applicability. It does not allow us to specify a priori the body shape and then determine the velocity potential or stream function that describes the flow around the particular body.

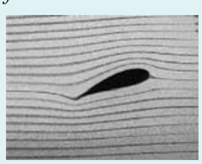
Determining the velocity potential or stream function for a given body shape is a much more complicated problem.

It is possible to extend the idea of superposition by considering a *distribution* of sources and sinks, or doublets, which when combined with a uniform flow can describe the flow around bodies of arbitrary shape. Techniques are available to determine the required distribution to give a prescribed body shape. Also, for plane potential flow problems it can be shown that complex variable theory (the use of real and imaginary numbers) can be effectively used to obtain solutions to a great variety of important flow problems. There are, of course, numerical techniques that can be used to solve not only plane two-dimensional problems, but the more general three-dimensional problems. Since potential flow is governed by Laplace's equation, any procedure that is available for solving this equation can be applied to the analysis of irrotational flow of frictionless fluids. Potential flow theory is an old and well-established discipline within the general field of fluid mechanics. The interested reader can find many detailed references on this subject, including Refs. 2, 3, 4, 5, and 6 given at the end of this chapter.

Potential flow solutions are always approximate because the fluid is assumed to be frictionless.

An important point to remember is that regardless of the particular technique used to obtain a solution to a potential flow problem, the solution remains approximate because of the fundamental assumption of a frictionless fluid. Thus, "exact" solutions based on potential flow theory represent, at best, only approximate solutions to real fluid problems. The applicability of potential flow theory to real fluid problems has been alluded to in a number of examples considered in the previous section. As a rule of thumb, potential flow theory will usually provide a reasonable approximation in those circumstances when we are dealing with a low viscosity fluid moving at a relatively high velocity, in regions of the flow field in which the flow is accelerating. Under these circumstances we generally find that the effect of viscosity is confined to the thin boundary layer that develops at a solid boundary. Outside the boundary layer the velocity distribution and the pressure distribution are closely approximated by the potential flow solution. However, in those regions of the flow field in which the flow is decelerating (for example, in the rearward portion of a bluff body or in the expanding region of a conduit), the pressure near a solid boundary will increase in the direction of flow. This so-called adverse pressure gradient can lead to flow separation, a phenomenon that causes dramatic changes in the flow field which are generally not accounted for by potential theory. However, as discussed in Chapter 9, in which boundary layer theory is developed, it is found that potential flow theory is used to obtain the appropriate pressure distribution that can then be combined with the viscous flow equations to obtain solutions near the boundary (and also to predict separation). The general differential equations that describe viscous fluid behavior and some simple solutions to these equations are considered in the remaining sections of this chapter.

V6.10 Potential flow



6.8 Viscous Flow

To incorporate viscous effects into the differential analysis of fluid motion we must return to the previously derived general equations of motion, Eqs. 6.50. Since these equations include both stresses and velocities, there are more unknowns than equations, and therefore before proceeding it is necessary to establish a relationship between the stresses and velocities.

6.8.1 Stress–Deformation Relationships

For incompressible Newtonian fluids it is known that the stresses are linearly related to the rates of deformation and can be expressed in Cartesian coordinates as (for normal stresses)

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \qquad \textbf{(6.125a)}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} \qquad \textbf{(6.125b)}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \qquad \textbf{(6.125c)}$$

(for shearing stresses)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \tag{6.125d}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{6.125e}$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \tag{6.125f}$$

where p is the pressure, the negative of the average of the three normal stresses; that is, $-p = (\frac{1}{3})(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$. For viscous fluids in motion the normal stresses are not necessarily the same in different directions, thus, the need to define the pressure as the average of the three normal stresses. For fluids at rest, or frictionless fluids, the normal stresses are equal in all directions. (We have made use of this fact in the chapter on fluid statics and in developing the equations for inviscid flow.) Detailed discussions of the development of these stress–velocity gradient relationships can be found in Refs. 3, 7, and 8. An important point to note is that whereas for elastic solids the stresses are linearly related to the deformation (or strain), for Newtonian fluids the stresses are linearly related to the rate of deformation (or rate of strain).

For Newtonian fluids, stresses are linearly related to the rate of strain.

In cylindrical polar coordinates the stresses for incompressible Newtonian fluids are expressed as (for normal stresses)

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r} \tag{6.126a}$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) \tag{6.126b}$$

$$\sigma_{zz} = -p + 2\mu\frac{\partial v_z}{\partial z} \tag{6.126c}$$

(for shearing stresses)

$$\tau_{r\theta} = \tau_{\theta r} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right] \tag{6.126d}$$

$$\tau_{\theta z} = \tau_{z\theta} = \mu\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \tag{6.126e}$$

$$\tau_{zr} = \tau_{rz} = \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) \tag{6.126f}$$

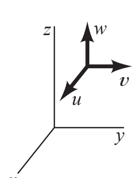
The double subscript has a meaning similar to that of stresses expressed in Cartesian coordinates—that is, the first subscript indicates the plane on which the stress acts, and the second subscript the direction. Thus, for example, σ_{rr} refers to a stress acting on a plane perpendicular to the radial direction and in the radial direction (thus a normal stress). Similarly, $\tau_{r\theta}$ refers to a stress acting on a plane perpendicular to the radial direction but in the tangential (θ direction) and is therefore a shearing stress.

6.8.2 The Navier–Stokes Equations

The stresses as defined in the preceding section can be substituted into the differential equations of motion (Eqs. 6.50) and simplified by using the continuity equation (Eq. 6.31) to obtain:

(x direction)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{6.127a}$$



(y direction)

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{6.127b}$$

(z direction)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad \textbf{(6.127c)}$$

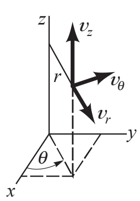
The Navier–Stokes equations are the basic differential equations describing the flow of Newtonian fluids.

where u, v, and w are the x, y, and z components of velocity as shown in the figure in the margin of the previous page. We have rearranged the equations so the acceleration terms are on the left side and the force terms are on the right. These equations are commonly called the ***Navier–Stokes equations,*** named in honor of the French mathematician L. M. H. Navier (1785–1836) and the English mechanician Sir G. G. Stokes (1819–1903), who were responsible for their formulation. These three equations of motion, when combined with the conservation of mass equation (Eq. 6.31), provide a complete mathematical description of the flow of incompressible Newtonian fluids. We have four equations and four unknowns (u, v, w, and p), and therefore the problem is "well-posed" in mathematical terms. Unfortunately, because of the general complexity of the Navier–Stokes equations (they are nonlinear, second-order, partial differential equations), they are not amenable to exact mathematical solutions except in a few instances. However, in those few instances in which solutions have been obtained and compared with experimental results, the results have been in close agreement. Thus, the Navier–Stokes equations are considered to be the governing differential equations of motion for incompressible Newtonian fluids.

In terms of cylindrical polar coordinates (see the figure in the margin), the Navier–Stokes equations can be written as

(r direction)

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial r} + \rho g_r + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right] \quad \textbf{(6.128a)}$$



(θ direction)

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right)$$
$$= -\frac{1}{r}\frac{\partial p}{\partial \theta} + \rho g_\theta + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] \quad \textbf{(6.128b)}$$

(z direction)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right)$$
$$= -\frac{\partial p}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] \quad \textbf{(6.128c)}$$

To provide a brief introduction to the use of the Navier–Stokes equations, a few of the simplest exact solutions are developed in the next section. Although these solutions will prove to be relatively simple, this is not the case in general. In fact, only a few other exact solutions have been obtained.

6.9 Some Simple Solutions for Laminar, Viscous, Incompressible Fluids

A principal difficulty in solving the Navier–Stokes equations is because of their nonlinearity arising from the convective acceleration terms (i.e., $u\,\partial u/\partial x$, $w\,\partial v/\partial z$, etc.). There are no general analytical schemes for solving nonlinear partial differential equations (e.g., superposition of solutions cannot be used), and each problem must be considered individually. For most practical flow problems, fluid particles do have accelerated motion as they move from one location to another in the flow field. Thus, the convective acceleration terms are usually important. However, there are a few special cases for which the convective acceleration vanishes because of the nature of the geometry of the flow system. In these cases exact solutions are often possible. The Navier–Stokes equations apply to both laminar and turbulent flow, but for turbulent flow each velocity component fluctuates randomly with respect to time and this added complication makes an analytical solution intractable. Thus, the exact solutions referred to are for laminar flows in which the velocity is either independent of time (steady flow) or dependent on time (unsteady flow) in a well-defined manner.

6.9.1 Steady, Laminar Flow between Fixed Parallel Plates

An exact solution can be obtained for steady laminar flow between fixed parallel plates.

We first consider flow between the two horizontal, infinite parallel plates of Fig. 6.31*a*. For this geometry the fluid particles move in the x direction parallel to the plates, and there is no velocity in the y or z direction—that is, $v = 0$ and $w = 0$. In this case it follows from the continuity equation (Eq. 6.31) that $\partial u/\partial x = 0$. Furthermore, there would be no variation of u in the z direction for infinite plates, and for steady flow $\partial u/\partial t = 0$ so that $u = u(y)$. If these conditions are used in the Navier–Stokes equations (Eqs. 6.127), they reduce to

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(6.129)}$$

$$0 = -\frac{\partial p}{\partial y} - \rho g \qquad \textbf{(6.130)}$$

$$0 = -\frac{\partial p}{\partial z} \qquad \textbf{(6.131)}$$

where we have set $g_x = 0$, $g_y = -g$, and $g_z = 0$. That is, the y axis points up. We see that for this particular problem the Navier–Stokes equations reduce to some rather simple equations.

Equations 6.130 and 6.131 can be integrated to yield

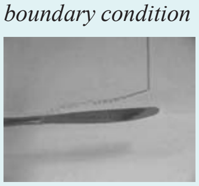
$$p = -\rho g y + f_1(x) \qquad \textbf{(6.132)}$$

which shows that the pressure varies hydrostatically in the y direction. Equation 6.129, rewritten as

$$\frac{d^2 u}{dy^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

can be integrated to give

V6.11 No-slip boundary condition

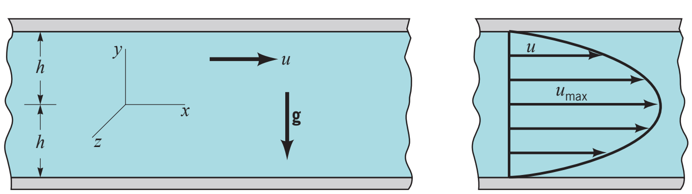


$$\frac{du}{dy} = \frac{1}{\mu}\left(\frac{\partial p}{\partial x}\right)y + c_1$$

and integrated again to yield

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)y^2 + c_1 y + c_2 \qquad \textbf{(6.133)}$$

Note that for this simple flow the pressure gradient, $\partial p/\partial x$, is treated as constant as far as the integration is concerned, since (as shown in Eq. 6.132) it is not a function of y. The two constants c_1 and c_2 must be determined from the boundary conditions. For example, if the two plates are



(*a*) (*b*)

■ **FIGURE 6.31** **The viscous flow between parallel plates: (*a*) coordinate system and notation used in analysis; (*b*) parabolic velocity distribution for flow between parallel fixed plates.**

fixed, then $u = 0$ for $y = \pm h$ (because of the no-slip condition for viscous fluids). To satisfy this condition $c_1 = 0$ and

$$c_2 = -\frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)h^2$$

Thus, the velocity distribution becomes

$$u = \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2) \qquad \textbf{(6.134)}$$

Equation 6.134 shows that the velocity profile between the two fixed plates is parabolic as illustrated in Fig. 6.31*b*.

The volume rate of flow, q, passing between the plates (for a unit width in the z direction) is obtained from the relationship

V6.12 Liquid–liquid no-slip

$$q = \int_{-h}^{h} u\,dy = \int_{-h}^{h} \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - h^2)\,dy$$

or

$$q = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right) \qquad \textbf{(6.135)}$$

The pressure gradient $\partial p/\partial x$ is negative, since the pressure decreases in the direction of flow. If we let Δp represent the pressure *drop* between two points a distance ℓ apart, then

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial x}$$

and Eq. 6.135 can be expressed as

$$q = \frac{2h^3 \Delta p}{3\mu\ell} \qquad \textbf{(6.136)}$$

The flow is proportional to the pressure gradient, inversely proportional to the viscosity, and strongly dependent ($\sim h^3$) on the gap width. In terms of the mean velocity, V, where $V = q/2h$, Eq. 6.136 becomes

$$V = \frac{h^2 \Delta p}{3\mu\ell} \qquad \textbf{(6.137)}$$

Equations 6.136 and 6.137 provide convenient relationships for relating the pressure drop along a parallel-plate channel and the rate of flow or mean velocity. The maximum velocity, u_{max}, occurs midway ($y = 0$) between the two plates, as shown in Fig. 6.31*b*, so that from Eq. 6.134

$$u_{max} = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right)$$

or

$$u_{max} = \tfrac{3}{2}V \qquad \textbf{(6.138)}$$

The Navier–Stokes equations provide detailed flow characteristics for laminar flow between fixed parallel plates.

The details of the steady laminar flow between infinite parallel plates are completely predicted by this solution to the Navier–Stokes equations. For example, if the pressure gradient, viscosity, and plate spacing are specified, then from Eq. 6.134 the velocity profile can be determined, and from Eqs. 6.136 and 6.137 the corresponding flowrate and mean velocity determined. In addition, from Eq. 6.132 it follows that

$$f_1(x) = \left(\frac{\partial p}{\partial x}\right)x + p_0$$

where p_0 is a reference pressure at $x = y = 0$, and the pressure variation throughout the fluid can be obtained from

$$p = -\rho g y + \left(\frac{\partial p}{\partial x}\right)x + p_0 \qquad \textbf{(6.139)}$$

For a given fluid and reference pressure, p_0, the pressure at any point can be predicted. This relatively simple example of an exact solution illustrates the detailed information about the flow field which can be obtained. The flow will be laminar if the Reynolds number, $\text{Re} = \rho V(2h)/\mu$, remains below about 1400. For flow with larger Reynolds numbers the flow becomes turbulent and the preceding analysis is not valid since the flow field is complex, three-dimensional, and unsteady.

F l u i d s i n t h e N e w s

10 tons on 8 psi Place a golf ball on the end of a garden hose and then slowly turn the water on a small amount until the ball just barely lifts off the end of the hose, leaving a small gap between the ball and the hose. The ball is free to rotate. This is the idea behind the new "floating ball water fountains" developed in Finland. Massive, 10-ton, 6-ft-diameter stone spheres are supported by the pressure force of the water on the curved surface within a pedestal and rotate so easily that even a small child can change their direction of rotation. The key to the fountain design is the ability to grind and polish stone to an accuracy of a few thousandths of an inch. This allows the gap between the ball and its pedestal to be very small (on the order of 5/1000 in.) and the water flowrate correspondingly small (on the order of 5 gallons per minute). Due to the small gap, the flow in the gap is essentially that of *flow between parallel plates*. Although the sphere is very heavy, the pressure under the sphere within the pedestal needs to be only about 8 psi. (See Problem 6.88.)

6.9.2 Couette Flow

For a given flow geometry, the character and details of the flow are strongly dependent on the boundary conditions.

Another simple parallel-plate flow can be developed by fixing one plate and letting the other plate move with a constant velocity, U, as is illustrated in Fig. 6.32*a*. The Navier–Stokes equations reduce to the same form as those in the preceding section, and the solution for the pressure and velocity distribution are still given by Eqs. 6.132 and 6.133, respectively. However, for the moving plate problem the boundary conditions for the velocity are different. For this case we locate the origin of the coordinate system at the bottom plate and designate the distance between the two plates as b (see Fig. 6.32*a*). The two constants c_1 and c_2 in Eq. 6.133 can be determined from the boundary conditions, $u = 0$ at $y = 0$ and $u = U$ at $y = b$. It follows that

$$u = U\frac{y}{b} + \frac{1}{2\mu}\left(\frac{\partial p}{\partial x}\right)(y^2 - by) \tag{6.140}$$

or, in dimensionless form,

$$\frac{u}{U} = \frac{y}{b} - \frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right) \tag{6.141}$$

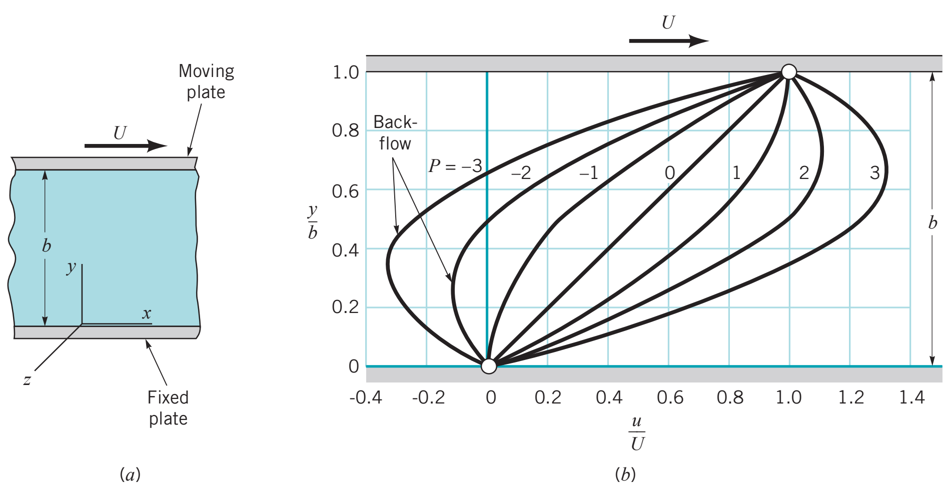
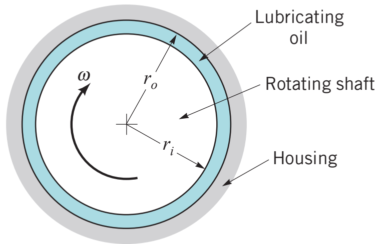


FIGURE 6.32 **The viscous flow between parallel plates with bottom plate fixed and upper plate moving (Couette flow): (*a*) coordinate system and notation used in analysis; (*b*) velocity distribution as a function of parameter, *P*, where $P = -(b^2/2\mu U)\ \partial p/\partial x$. (From Ref. 8, used by permission.)**



■ **FIGURE 6.33** **Flow in the narrow gap of a journal bearing.**

The actual velocity profile will depend on the dimensionless parameter

$$P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

Flow between parallel plates with one plate fixed and the other moving is called Couette flow.

Several profiles are shown in Fig. 6.32b. This type of flow is called ***Couette flow.***

The simplest type of Couette flow is one for which the pressure gradient is zero; that is, the fluid motion is caused by the fluid being dragged along by the moving boundary. In this case, with $\partial p/\partial x = 0$, Eq. 6.140 simply reduces to

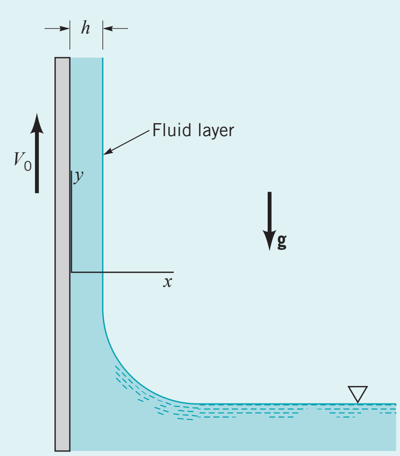
$$u = U\frac{y}{b} \qquad \textbf{(6.142)}$$

which indicates that the velocity varies linearly between the two plates as shown in Fig. 6.31b for $P = 0$. This situation would be approximated by the flow between closely spaced concentric cylinders in which one cylinder is fixed and the other cylinder rotates with a constant angular velocity, ω. As illustrated in Fig. 6.33, the flow in an unloaded journal bearing might be approximated by this simple Couette flow if the gap width is very small (i.e., $r_o - r_i \ll r_i$). In this case $U = r_i\omega$, $b = r_o - r_i$, and the shearing stress resisting the rotation of the shaft can be simply calculated as $\tau = \mu r_i\omega/(r_o - r_i)$. When the bearing is loaded (i.e., a force applied normal to the axis of rotation), the shaft will no longer remain concentric with the housing and the flow cannot be treated as flow between parallel boundaries. Such problems are dealt with in lubrication theory (see, for example, Ref. 9).

EXAMPLE 6.9 Plane Couette Flow

GIVEN A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0, as illustrated in Fig. E6.9a. Because of viscous forces the belt picks up a film of fluid of thickness h. Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.

FIND Use the Navier–Stokes equations to determine an expression for the average velocity of the fluid film as it is dragged up the belt.



■ **FIGURE E6.9a**

SOLUTION

Since the flow is assumed to be fully developed, the only velocity component is in the y direction (the v component) so that $u = w = 0$. It follows from the continuity equation that $\partial v/\partial y = 0$, and for steady flow $\partial v/\partial t = 0$, so that $v = v(x)$. Under these conditions the Navier–Stokes equations for the x direction (Eq. 6.127a) and the z direction (perpendicular to the paper) (Eq. 6.127c) simply reduce to

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial z} = 0$$

This result indicates that the pressure does not vary over a horizontal plane, and since the pressure on the surface of the film ($x = h$) is atmospheric, the pressure throughout the film must be atmospheric (or zero gage pressure). The equation of motion in the y direction (Eq. 6.127b) thus reduces to

$$0 = -\rho g + \mu \frac{d^2 v}{dx^2}$$

or

$$\frac{d^2 v}{dx^2} = \frac{\gamma}{\mu} \qquad \textbf{(1)}$$

Integration of Eq. 1 yields

$$\frac{dv}{dx} = \frac{\gamma}{\mu} x + c_1 \qquad \textbf{(2)}$$

On the film surface ($x = h$) we assume the shearing stress is zero—that is, the drag of the air on the film is negligible. The shearing stress at the free surface (or any interior parallel surface) is designated as τ_{xy}, where from Eq. 6.125d

$$\tau_{xy} = \mu\left(\frac{dv}{dx}\right)$$

Thus, if $\tau_{xy} = 0$ at $x = h$, it follows from Eq. 2 that

$$c_1 = -\frac{\gamma h}{\mu}$$

A second integration of Eq. 2 gives the velocity distribution in the film as

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + c_2$$

At the belt ($x = 0$) the fluid velocity must match the belt velocity, V_0, so that

$$c_2 = V_0$$

and the velocity distribution is therefore

$$v = \frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0 \qquad \textbf{(3)}$$

With the velocity distribution known we can determine the flowrate per unit width, q, from the relationship

$$q = \int_0^h v\, dx = \int_0^h \left(\frac{\gamma}{2\mu} x^2 - \frac{\gamma h}{\mu} x + V_0\right) dx$$

and thus

$$q = V_0 h - \frac{\gamma h^3}{3\mu}$$

The average film velocity, V (where $q = Vh$), is therefore

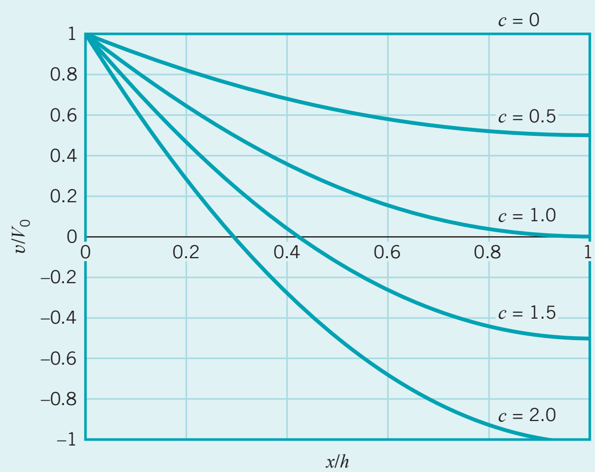
$$V = V_0 - \frac{\gamma h^2}{3\mu} \qquad \textbf{(Ans)}$$

COMMENT Equation (3) can be written in dimensionless form as

$$\frac{v}{V_0} = c\left(\frac{x}{h}\right)^2 - 2c\left(\frac{x}{h}\right) + 1$$

where $c = \gamma h^2/2\mu V_0$. This velocity profile is shown in Fig. E6.9b. Note that even though the belt is moving upward, for $c > 1$ (e.g., for fluids with small enough viscosity or with a small enough belt speed) there are portions of the fluid that flow downward (as indicated by $v/V_0 < 0$).

It is interesting to note from this result that there will be a net upward flow of liquid (positive V) only if $V_0 > \gamma h^2/3\mu$. It takes a relatively large belt speed to lift a small viscosity fluid.



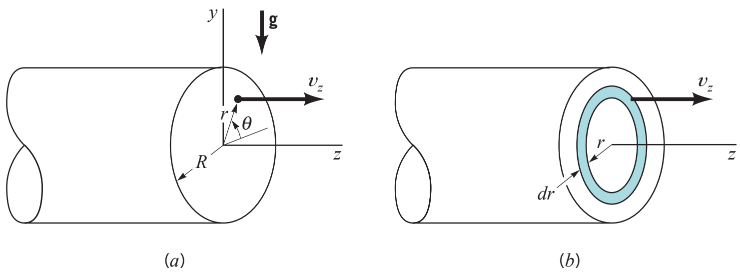
■ **FIGURE E6.9*b***

6.9.3 Steady, Laminar Flow in Circular Tubes

An exact solution can be obtained for steady, incompressible, laminar flow in circular tubes.

Probably the best known exact solution to the Navier–Stokes equations is for steady, incompressible, laminar flow through a straight circular tube of constant cross section. This type of flow is commonly called *Hagen–Poiseuille flow*, or simply *Poiseuille flow.* It is named in honor of J. L. Poiseuille (1799–1869), a French physician, and G. H. L. Hagen (1797–1884), a German hydraulic engineer. Poiseuille was interested in blood flow through capillaries and deduced experimentally the resistance laws for laminar flow through circular tubes. Hagen's investigation of flow in tubes was also experimental. It was actually after the work of Hagen and Poiseuille that the theoretical results presented in this section were determined, but their names are commonly associated with the solution of this problem.

Consider the flow through a horizontal circular tube of radius R as is shown in Fig. 6.34a. Because of the cylindrical geometry it is convenient to use cylindrical coordinates. We assume that the flow is parallel to the walls so that $v_r = 0$ and $v_\theta = 0$, and from the continuity equation (6.34) $\partial v_z/\partial z = 0$. Also, for steady, axisymmetric flow, v_z is not a function of t or θ so the velocity, v_z,



■ **FIGURE 6.34** **The viscous flow in a horizontal, circular tube: (*a*) coordinate system and notation used in analysis; (*b*) flow through differential annular ring.**

is only a function of the radial position within the tube—that is, $v_z = v_z(r)$. Under these conditions the Navier–Stokes equations (Eqs. 6.128) reduce to

$$0 = -\rho g \sin\theta - \frac{\partial p}{\partial r} \tag{6.143}$$

$$0 = -\rho g \cos\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} \tag{6.144}$$

$$0 = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right] \tag{6.145}$$

where we have used the relationships $g_r = -g\sin\theta$ and $g_\theta = -g\cos\theta$ (with θ measured from the horizontal plane).

Equations 6.143 and 6.144 can be integrated to give

$$p = -\rho g(r\sin\theta) + f_1(z)$$

or

$$p = -\rho g y + f_1(z) \tag{6.146}$$

V6.13 Laminar flow

Equation 6.146 indicates that the pressure is hydrostatically distributed at any particular cross section, and the z component of the pressure gradient, $\partial p/\partial z$, is not a function of r or θ.

The equation of motion in the z direction (Eq. 6.145) can be written in the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) = \frac{1}{\mu}\frac{\partial p}{\partial z}$$

and integrated (using the fact that $\partial p/\partial z =$ constant) to give

$$r\frac{\partial v_z}{\partial r} = \frac{1}{2\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1$$

Integrating again we obtain

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)r^2 + c_1 \ln r + c_2 \tag{6.147}$$

Since we wish v_z to be finite at the center of the tube ($r = 0$), it follows that $c_1 = 0$ [since $\ln(0) = -\infty$]. At the wall ($r = R$) the velocity must be zero so that

$$c_2 = -\frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)R^2$$

and the velocity distribution becomes

The velocity distribution is parabolic for steady, laminar flow in circular tubes.

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2) \tag{6.148}$$

Thus, at any cross section the velocity distribution is parabolic.

To obtain a relationship between the volume rate of flow, Q, passing through the tube and the pressure gradient, we consider the flow through the differential, washer-shaped ring of Fig. 6.34*b*. Since v_z is constant on this ring, the volume rate of flow through the differential area $dA = (2\pi r)\,dr$ is

$$dQ = v_z(2\pi r)\,dr$$

and therefore

$$Q = 2\pi \int_0^R v_z r \, dr \tag{6.149}$$

Equation 6.148 for v_z can be substituted into Eq. 6.149, and the resulting equation integrated to yield

V6.14 Complex pipe flow

$$Q = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) \tag{6.150}$$

This relationship can be expressed in terms of the pressure *drop*, Δp, which occurs over a length, ℓ, along the tube, since

$$\frac{\Delta p}{\ell} = -\frac{\partial p}{\partial z}$$

and therefore

$$Q = \frac{\pi R^4 \Delta p}{8\mu \ell} \tag{6.151}$$

Poiseuille's law relates pressure drop and flowrate for steady, laminar flow in circular tubes.

For a given pressure drop per unit length, the volume rate of flow is inversely proportional to the viscosity and proportional to the tube radius to the fourth power. A doubling of the tube radius produces a 16-fold increase in flow! Equation 6.151 is commonly called ***Poiseuille's law.***

In terms of the mean velocity, V, where $V = Q/\pi R^2$, Eq. 6.151 becomes

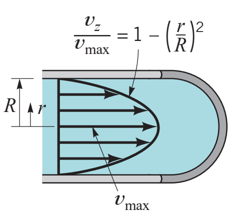
$$V = \frac{R^2 \Delta p}{8\mu \ell} \tag{6.152}$$

The maximum velocity $v_{\max}$ occurs at the center of the tube, where from Eq. 6.148

$$v_{\max} = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu \ell} \tag{6.153}$$

so that

$$v_{\max} = 2V$$



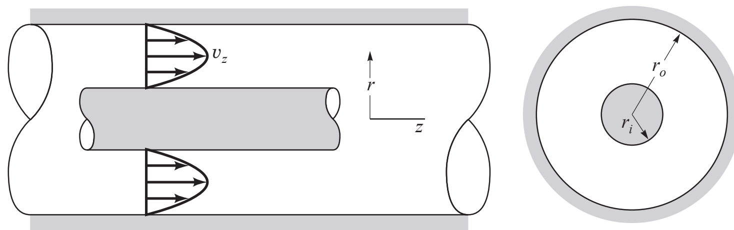
The velocity distribution, as shown by the figure in the margin, can be written in terms of $v_{\max}$ as

$$\frac{v_z}{v_{\max}} = 1 - \left(\frac{r}{R}\right)^2 \tag{6.154}$$

As was true for the similar case of flow between parallel plates (sometimes referred to as *plane Poiseuille flow*), a very detailed description of the pressure and velocity distribution in tube flow results from this solution to the Navier–Stokes equations. Numerous experiments performed to substantiate the theoretical results show that the theory and experiment are in agreement for the laminar flow of Newtonian fluids in circular tubes or pipes. In general, the flow remains laminar for Reynolds numbers, Re $= \rho V(2R)/\mu$, below 2100. Turbulent flow in tubes is considered in Chapter 8.

F l u i d s i n t h e N e w s

Poiseuille's law revisited Poiseuille's law governing *laminar flow* of fluids in tubes has an unusual history. It was developed in 1842 by a French physician, J. L. M. Poiseuille, who was interested in the flow of blood in capillaries. Poiseuille, through a series of carefully conducted experiments using water flowing through very small tubes, arrived at the formula, $Q = K\Delta p\, D^4/\ell$. In this formula Q is the flowrate, K an empirical constant, Δp the pressure drop over the length ℓ, and D the tube diameter. Another formula was given for the value of K as a function of the water temperature. It was not until the concept of viscosity was introduced at a later date that Poiseuille's law was derived mathematically and the constant K found to be equal to $\pi/8\mu$, where μ is the fluid viscosity. The experiments by Poiseuille have long been admired for their accuracy and completeness considering the laboratory instrumentation available in the mid nineteenth century.



■ FIGURE 6.35 The viscous flow through an annulus.

6.9.4 Steady, Axial, Laminar Flow in an Annulus

An exact solution can be obtained for axial flow in the annular space between two fixed, concentric cylinders.

The differential equations (Eqs. 6.143, 6.144, 6.145) used in the preceding section for flow in a tube also apply to the axial flow in the annular space between two fixed, concentric cylinders (Fig. 6.35). Equation 6.147 for the velocity distribution still applies, but for the stationary annulus the boundary conditions become $v_z = 0$ at $r = r_o$ and $v_z = 0$ for $r = r_i$. With these two conditions the constants c_1 and c_2 in Eq. 6.147 can be determined and the velocity distribution becomes

$$v_z = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)\left[r^2 - r_o^2 + \frac{r_i^2 - r_o^2}{\ln(r_o/r_i)}\ln\frac{r}{r_o}\right] \tag{6.155}$$

The corresponding volume rate of flow is

$$Q = \int_{r_i}^{r_o} v_z(2\pi r)\,dr = -\frac{\pi}{8\mu}\left(\frac{\partial p}{\partial z}\right)\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]$$

or in terms of the pressure drop, Δp, in length ℓ of the annulus

$$Q = \frac{\pi\Delta p}{8\mu\ell}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right] \tag{6.156}$$

The velocity at any radial location within the annular space can be obtained from Eq. 6.155. The maximum velocity occurs at the radius $r = r_m$ where $\partial v_z/\partial r = 0$. Thus,

$$r_m = \left[\frac{r_o^2 - r_i^2}{2\ln(r_o/r_i)}\right]^{1/2} \tag{6.157}$$

An inspection of this result shows that the maximum velocity does not occur at the midpoint of the annular space, but rather it occurs nearer the inner cylinder. The specific location depends on r_o and r_i.

These results for flow through an annulus are valid only if the flow is laminar. A criterion based on the conventional Reynolds number (which is defined in terms of the tube diameter) cannot be directly applied to the annulus, since there are really "two" diameters involved. For tube cross sections other than simple circular tubes it is common practice to use an "effective" diameter, termed the *hydraulic diameter*, D_h, which is defined as

$$D_h = \frac{4 \times \text{cross-sectional area}}{\text{wetted perimeter}}$$

The wetted perimeter is the perimeter in contact with the fluid. For an annulus

$$D_h = \frac{4\pi(r_o^2 - r_i^2)}{2\pi(r_o + r_i)} = 2(r_o - r_i)$$

In terms of the hydraulic diameter, the Reynolds number is $\text{Re} = \rho D_h V/\mu$ (where $V = Q/$cross-sectional area), and it is commonly assumed that if this Reynolds number remains below 2100 the flow will be laminar. A further discussion of the concept of the hydraulic diameter as it applies to other noncircular cross sections is given in Section 8.4.3.

EXAMPLE 9.10 Laminar Flow in an Annulus

GIVEN A viscous liquid ($\rho = 1.18 \times 10^3$ kg/m^3; $\mu = 0.0045$ N·s/m^2) flows at a rate of 12 ml/s through a horizontal, 4-mm-diameter tube.

FIND (a) Determine the pressure drop along a l-m length of the tube which is far from the tube entrance so that the only component of velocity is parallel to the tube axis. (b) If a 2-mm-diameter rod is placed in the 4-mm-diameter tube to form a symmetric annulus, what is the pressure drop along a l-m length if the flowrate remains the same as in part (a)?

SOLUTION

(a) We first calculate the Reynolds number, Re, to determine whether or not the flow is laminar. With the diameter $D = 4$ mm $= 0.004$ m, the mean velocity is

$$V = \frac{Q}{(\pi/4)D^2} = \frac{(12 \text{ ml/s})(10^{-6} \text{ m}^3/\text{ml})}{(\pi/4)(0.004 \text{ m})^2} = 0.955 \text{ m/s}$$

and, therefore,

$$\text{Re} = \frac{\rho V D}{\mu} = \frac{(1.18 \times 10^3 \text{ kg/m}^3)(0.955 \text{ m/s})(0.004 \text{ m})}{0.0045 \text{ N}\cdot\text{s/m}^2} = 1000$$

Since the Reynolds number is well below the critical value of 2100 we can safely assume that the flow is laminar. Thus, we can apply Eq. 6.151, which gives for the pressure drop

$$\Delta p = \frac{8\mu \ell Q}{\pi R^4} = \frac{8(0.0045 \text{ N}\cdot\text{s/m}^2)(1 \text{ m})(12 \times 10^{-6} \text{ m}^3/\text{s})}{\pi(0.002 \text{ m})^4} = 8.59 \text{ kPa} \quad \textbf{(Ans)}$$

(b) For flow in the annulus with an outer radius $r_o = 0.002$ m and an inner radius $r_i = 0.001$ m, the mean velocity is

$$V = \frac{Q}{\pi(r_o^2 - r_i^2)} = \frac{12 \times 10^{-6} \text{ m}^3/\text{s}}{(\pi)[(0.002 \text{ m})^2 - (0.001 \text{ m})^2]} = 1.27 \text{ m/s}$$

and the Reynolds number [based on the hydraulic diameter, $D_h = 2(r_o - r_i) = 2(0.002 \text{ m} - 0.001 \text{ m}) = 0.002$ m] is

$$\text{Re} = \frac{\rho D_h V}{\mu} = \frac{(1.18 \times 10^3 \text{ kg/m}^3)\,(0.002 \text{ m})\,(1.27 \text{ m/s})}{0.0045 \text{ N}\cdot\text{s/m}^2} = 666$$

This value is also well below 2100 so the flow in the annulus should also be laminar. From Eq. 6.156,

$$\Delta p = \frac{8\mu \ell Q}{\pi}\left[r_o^4 - r_i^4 - \frac{(r_o^2 - r_i^2)^2}{\ln(r_o/r_i)}\right]^{-1}$$

so that

$$\Delta p = \frac{8(0.0045 \text{ N}\cdot\text{s/m}^2)(1 \text{ m})(12 \times 10^{-6} \text{ m}^3/\text{s})}{\pi} \times \left\{(0.002 \text{ m})^4 - (0.001 \text{ m})^4 - \frac{[(0.002 \text{ m})^2 - (0.001 \text{ m})^2]^2}{\ln(0.002 \text{ m}/0.001 \text{ m})}\right\}^{-1} = 68.2 \text{ kPa} \quad \textbf{(Ans)}$$

COMMENTS The pressure drop in the annulus is much larger than that of the tube. This is not a surprising result, since to maintain the same flow in the annulus as that in the open tube, the average velocity must be larger (the cross-sectional area is smaller) and the pressure difference along the annulus must overcome the shearing stresses that develop along both an inner and an outer wall.

By repeating the calculations for various radius ratios, r_i/r_o, the results shown in Fig. E6.10 are obtained. It is seen that the pressure drop ratio, $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}}$ (i.e., the pressure drop in the annulus compared to that in a tube with a radius equal to the outer radius of the annulus, r_o), is a strong function of the radius ratio. Even an annulus with a very small inner radius will have a pressure drop significantly larger than that of a tube. For example, if the inner radius is only 1/100 of the outer radius, $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}} = 1.28$. As shown in the figure, for larger inner radii, the pressure drop ratio is much larger [i.e., $\Delta p_{\text{annulus}}/\Delta p_{\text{tube}} = 7.94$ for $r_i/r_o = 0.50$ as in part (b) of this example].

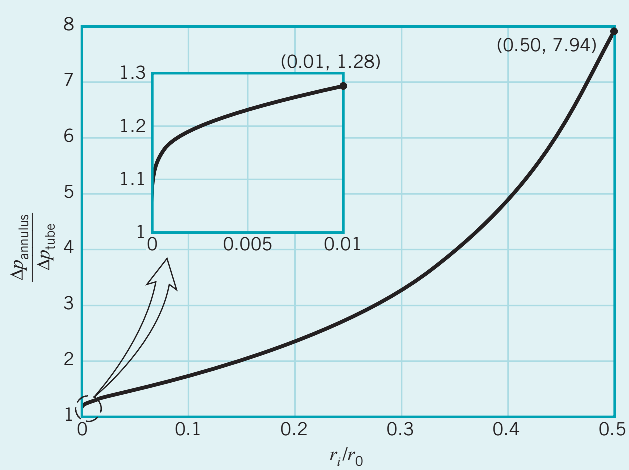


FIGURE E6.10

6.10 Other Aspects of Differential Analysis

In this chapter the basic differential equations that govern the flow of fluids have been developed. The Navier–Stokes equations, which can be compactly expressed in vector notation as

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \boldsymbol{\nabla}\mathbf{V}\right) = -\boldsymbol{\nabla} p + \rho \mathbf{g} + \mu \nabla^2 \mathbf{V} \tag{6.158}$$

along with the continuity equation

$$\nabla \cdot \mathbf{V} = 0 \tag{6.159}$$

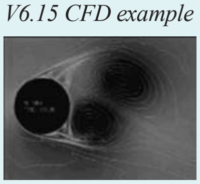
are the general equations of motion for incompressible Newtonian fluids. Although we have restricted our attention to incompressible fluids, these equations can be readily extended to include compressible fluids. It is well beyond the scope of this introductory text to consider in depth the variety of analytical and numerical techniques that can be used to obtain both exact and approximate solutions to the Navier–Stokes equations. Students, however, should be aware of the existence of these very general equations, which are frequently used as the basis for many advanced analyses of fluid motion. A few relatively simple solutions have been obtained and discussed in this chapter to indicate the type of detailed flow information that can be obtained by using differential analysis. However, it is hoped that the relative ease with which these solutions were obtained does not give the false impression that solutions to the Navier–Stokes equations are readily available. This is certainly not true, and as previously mentioned there are actually very few practical fluid flow problems that can be solved by using an exact analytical approach. In fact, there are no known analytical solutions to Eq. 6.158 for flow past any object such as a sphere, cube, or airplane.

Very few practical fluid flow problems can be solved using an exact analytical approach.

Because of the difficulty in solving the Navier–Stokes equations, much attention has been given to various types of approximate solutions. For example, if the viscosity is set equal to zero, the Navier–Stokes equations reduce to Euler's equations. Thus, the frictionless fluid solutions discussed previously are actually approximate solutions to the Navier–Stokes equations. At the other extreme, for problems involving slowly moving fluids, viscous effects may be dominant and the nonlinear (convective) acceleration terms can be neglected. This assumption greatly simplifies the analysis, since the equations now become linear. There are numerous analytical solutions to these "*slow flow*" or "*creeping flow*" problems. Another broad class of approximate solutions is concerned with flow in the very thin boundary layer. L. Prandtl showed in 1904 how the Navier–Stokes equations could be simplified to study flow in boundary layers. Such "boundary layer solutions" play a very important role in the study of fluid mechanics. A further discussion of boundary layers is given in Chapter 9.

6.10.1 Numerical Methods

V6.15 CFD example



Numerical methods using digital computers are, of course, commonly utilized to solve a wide variety of flow problems. As discussed previously, although the differential equations that govern the flow of Newtonian fluids [the Navier–Stokes equations (6.158)] were derived many years ago, there are few known analytical solutions to them. With the advent of high-speed digital computers it has become possible to obtain numerical solutions to these (and other fluid mechanics) equations for many different types of problems. A brief introduction to computational fluid dynamics (CFD) is given in Appendix A.

Access to a program called FlowLab is available with this textbook. FlowLab is an educational version of a commercial CFD program. The backbone of FlowLab is the Fluent CFD package, which was used to create the numerical animations of flow past a spinning football referenced at the beginning of the chapter (V6.1 and V6.2). FlowLab provides a virtual laboratory for fluids experiments that makes use of the power of CFD, but with a student-friendly interface. Chapters 7–9 contain fluids problems that require the use of FlowLab to obtain the solutions.

F l u i d s i n t h e N e w s

Fluids in the Academy Awards A computer science professor at Stanford University and his colleagues were awarded a Scientific and Technical Academy Award for applying the Navier–Stokes equations for use in Hollywood movies. These researchers make use of computational algorithms to numerically solve the Navier–Stokes equations (also termed computational fluid dynamics, or CFD) and simulate complex liquid flows. The realism of the simulations has found application in the entertainment industry. Movie producers have used the power of these numerical tools to simulate flows from ocean waves in "Pirates of the Caribbean" to lava flows in the final duel in "Star Wars: Revenge of the Sith." Therefore, even Hollywood has recognized the usefulness of CFD.

6.11 Chapter Summary and Study Guide

volumetric dilatation rate
vorticity
irrotational flow
continuity equation
stream function
Euler's equations of motion
ideal fluid
Bernoulli equation
velocity potential
potential flow
equipotential lines
flow net
uniform flow
source and sink
vortex
circulation
doublet
method of superposition
half-body
Rankine oval
Navier–Stokes equations
Couette flow
Poiseuille's law

Differential analysis of fluid flow is concerned with the development of concepts and techniques that can be used to provide a detailed, point by point, description of a flow field. Concepts related to the motion and deformation of a fluid element are introduced, including the Eulerian method for describing the velocity and acceleration of fluid particles. Linear deformation and angular deformation of a fluid element are described through the use of flow characteristics such as the volumetric dilatation rate, rate of angular deformation, and vorticity. The differential form of the conservation of mass equation (continuity equation) is derived in both rectangular and cylindrical polar coordinates.

Use of the stream function for the study of steady, incompressible, plane, two-dimensional flow is introduced. The general equations of motion are developed, and for inviscid flow these equations are reduced to the simpler Euler equations of motion. The Euler equations are integrated to give the Bernoulli equation, and the concept of irrotational flow is introduced. Use of the velocity potential for describing irrotational flow is considered in detail, and several basic velocity potentials are described, including those for a uniform flow, source or sink, vortex, and doublet. The technique of using various combinations of these basic velocity potentials, by superposition, to form new potentials is described. Flows around a half-body, a Rankine oval, and around a circular cylinder are obtained using this superposition technique.

Basic differential equations describing incompressible, viscous flow (the Navier–Stokes equations) are introduced. Several relatively simple solutions for steady, viscous, laminar flow between parallel plates and through circular tubes are included.

The following checklist provides a study guide for this chapter. When your study of the entire chapter and end-of-chapter exercises has been completed you should be able to

- write out meanings of the terms listed here in the margin and understand each of the related concepts. These terms are particularly important and are set in ***italic bold, and color*** type in the text.
- determine the acceleration of a fluid particle, given the equation for the velocity field.
- determine the volumetric dilatation rate, vorticity, and rate of angular deformation for a fluid element, given the equation for the velocity field.
- show that a given velocity field satisfies the continuity equation.
- use the concept of the stream function to describe a flow field.
- use the concept of the velocity potential to describe a flow field.
- use superposition of basic velocity potentials to describe simple potential flow fields.
- use the Navier–Stokes equations to determine the detailed flow characteristics of incompressible, steady, laminar, viscous flow between parallel plates and through circular tubes.

Some of the important equations in this chapter are:

Acceleration of fluid particle
$$\mathbf{a} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} \qquad \textbf{(6.2)}$$

Vorticity
$$\zeta = 2\,\boldsymbol{\omega} = \nabla \times \mathbf{V} \qquad \textbf{(6.17)}$$

Conservation of mass
$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \textbf{(6.27)}$$

Stream function	$u = \dfrac{\partial \psi}{\partial y} \qquad v = -\dfrac{\partial \psi}{\partial x}$	**(6.37)**
Euler's equations of motion	$\rho g_x - \dfrac{\partial p}{\partial x} = \rho\left(\dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z}\right)$	**(6.51a)**
	$\rho g_y - \dfrac{\partial p}{\partial y} = \rho\left(\dfrac{\partial v}{\partial t} + u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} + w\dfrac{\partial v}{\partial z}\right)$	**(6.51b)**
	$\rho g_z - \dfrac{\partial p}{\partial z} = \rho\left(\dfrac{\partial w}{\partial t} + u\dfrac{\partial w}{\partial x} + v\dfrac{\partial w}{\partial y} + w\dfrac{\partial w}{\partial z}\right)$	**(6.51c)**
Velocity potential	$\mathbf{V} = \nabla\phi$	**(6.65)**
Laplace's equation	$\nabla^2\phi = 0$	**(6.66)**
Uniform potential flow	$\phi = U(x\cos\alpha + y\sin\alpha) \quad \psi = U(y\cos\alpha - x\sin\alpha) \quad u = U\cos\alpha,\ v = U\sin\alpha$	
Source and sink	$\phi = \dfrac{m}{2\pi}\ln r \quad \psi = \dfrac{m}{2\pi}\theta \quad v_r = \dfrac{m}{2\pi r},\ v_\theta = 0$	
Vortex	$\phi = \dfrac{\Gamma}{2\pi}\theta \quad \psi = -\dfrac{\Gamma}{2\pi}\ln r \quad v_r = 0,\ v_\theta = \dfrac{\Gamma}{2\pi r}$	
Doublet	$\phi = \dfrac{K\cos\theta}{r} \quad \psi = -\dfrac{K\sin\theta}{r} \quad v_r = -\dfrac{K\cos\theta}{r^2},\ v_\theta = -\dfrac{K\cos\theta}{r^2}$	

The Navier–Stokes equations

(x direction)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad \textbf{(6.127a)}$$

(y direction)

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad \textbf{(6.127b)}$$

(z direction)

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \quad \textbf{(6.127c)}$$

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Review Problems

Go to Appendix G for a set of review problems with answers. Detailed solutions can be found in *Student Solution Manual and Study Guide for Fundamentals of Fluid Mechanics*, by Munson et al. (© 2009 John Wiley and Sons, Inc.).

Problems

Note: Unless otherwise indicated, use the values of fluid properties found in the tables on the inside of the front cover. Problems designated with an (*) are intended to be solved with the aid of a programmable calculator or a computer. Problems designated with a (†) are "open-ended" problems and require critical thinking in that to work them one must make various assumptions and provide the necessary data. There is not a unique answer to these problems.

Answers to the even-numbered problems are listed at the end of the book. Access to the videos that accompany problems can be obtained through the book's web site, www.wiley.com/college/munson. The lab-type problems can also be accessed on this web site.

Section 6.1 Fluid Element Kinematics

6.1 Obtain a photograph/image of a situation in which a fluid is undergoing angular deformation. Print this photo and write a brief paragraph that describes the situation involved.

6.2 The velocity in a certain two-dimensional flow field is given by the equation

$$\mathbf{V} = 2xt\hat{\mathbf{i}} - 2yt\hat{\mathbf{j}}$$

where the velocity is in ft/s when x, y, and t are in feet and seconds, respectively. Determine expressions for the local and convective components of acceleration in the x and y directions. What is the magnitude and direction of the velocity and the acceleration at the point $x = y = 2$ ft at the time $t = 0$?

6.3 The velocity in a certain flow field is given by the equation

$$\mathbf{V} = x\hat{\mathbf{i}} + x^2z\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

Determine the expressions for the three rectangular components of acceleration.

6.4 The three components of velocity in a flow field are given by

$$u = x^2 + y^2 + z^2$$
$$v = xy + yz + z^2$$
$$w = -3xz - z^2/2 + 4$$

(a) Determine the volumetric dilatation rate and interpret the results. **(b)** Determine an expression for the rotation vector. Is this an irrotational flow field?

6.5 Determine the vorticity field for the following velocity vector:

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

6.6 Determine an expression for the vorticity of the flow field described by

$$\mathbf{V} = -xy^3\hat{\mathbf{i}} + y^4\hat{\mathbf{j}}$$

Is the flow irrotational?

6.7 A one-dimensional flow is described by the velocity field

$$u = ay + by^2$$
$$v = w = 0$$

where a and b are constants. Is the flow irrotational? For what combination of constants (if any) will the rate of angular deformation as given by Eq. 6.18 be zero?

6.8 For a certain incompressible, two-dimensional flow field the velocity component in the y direction is given by the equation

$$v = 3xy + x^2y$$

Determine the velocity component in the x direction so that the volumetric dilatation rate is zero

6.9 An incompressible viscous fluid is placed between two large parallel plates as shown in Fig. P6.9. The bottom plate is fixed and the upper plate moves with a constant velocity, U. For these conditions the velocity distribution between the plates is linear and can be expressed as

$$u = U\frac{y}{b}$$

Determine: **(a)** the volumetric dilatation rate, **(b)** the rotation vector, **(c)** the vorticity, and **(d)** the rate of angular deformation.

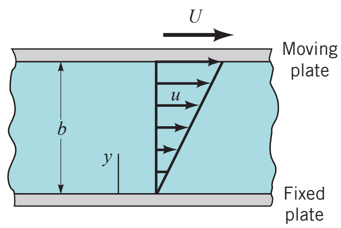


FIGURE P6.9

6.10 A viscous fluid is contained in the space between concentric cylinders. The inner wall is fixed, and the outer wall rotates with an angular velocity ω. (See Fig. P6.10a and **Video V6.3.**) Assume that the velocity distribution in the gap is linear as illustrated in Fig. P6.10b. For the small rectangular element shown in Fig. P6.10b, determine the rate of change of the right angle γ due to the fluid motion. Express your answer in terms of r_0, r_i, and ω.

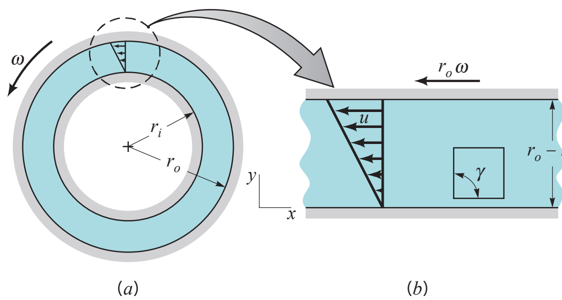


FIGURE P6.10

Section 6.2 Conservation of Mass

6.11 Obtain a photograph/image of a situation in which streamlines indicate a feature of the flow field. Print this photo and write a brief paragraph that describes the situation involved.

6.12 Verify that the stream function in cylindrical coordinates satisfies the continuity equation.

6.13 For a certain incompressible flow field it is suggested that the velocity components are given by the equations

$$u = 2xy \quad v = -x^2y \quad w = 0$$

Is this a physically possible flow field? Explain.

6.14 The velocity components of an incompressible, two-dimensional velocity field are given by the equations

$$u = y^2 - x(1 + x)$$
$$v = y(2x + 1)$$

Show that the flow is irrotational and satisfies conservation of mass.

6.15 For each of the following stream functions, with units of m^2/s, determine the magnitude and the angle the velocity vector makes with the x axis at $x = 1$ m, $y = 2$ m. Locate any stagnation points in the flow field.
(a) $\psi = xy$
(b) $\psi = -2x^2 + y$

6.16 The stream function for an incompressible, two-dimensional flow field is

$$\psi = ay - by^3$$

where a and b are constants. Is this an irrotational flow? Explain.

6.17 The stream function for an incompressible, two-dimensional flow field is

$$\psi = ay^2 - bx$$

where a and b are constants. Is this an irrotational flow? Explain.

6.18 The velocity components for an incompressible, plane flow are

$$v_r = Ar^{-1} + Br^{-2}\cos\theta$$
$$v_\theta = Br^{-2}\sin\theta$$

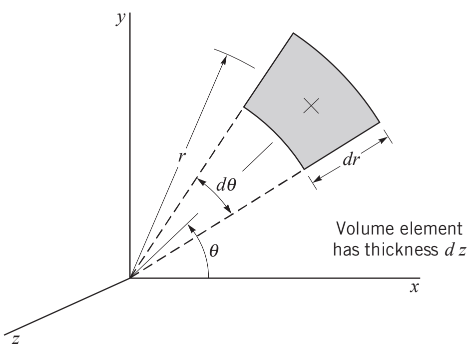
where A and B are constants. Determine the corresponding stream function.

6.19 For a certain two-dimensional flow field

$$u = 0$$
$$v = V$$

(a) What are the corresponding radial and tangential velocity components? **(b)** Determine the corresponding stream function expressed in Cartesian coordinates and in cylindrical polar coordinates.

6.20 Make use of the control volume shown in Fig. P6.20 to derive the continuity equation in cylindrical coordinates (Eq. 6.33 in text).

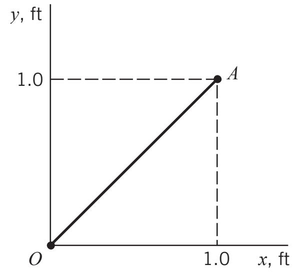


■ **FIGURE P6.20**

6.21 A two-dimensional, incompressible flow is given by $u = -y$ and $v = x$. Show that the streamline passing through the point $x = 10$ and $y = 0$ is a circle centered at the origin.

6.22 In a certain steady, two-dimensional flow field the fluid density varies linearly with respect to the coordinate x; that is, $\rho = Ax$ where A is a constant. If the x component of velocity u is given by the equation $u = y$, determine an expression for v.

6.23 In a two-dimensional, incompressible flow field, the x component of velocity is given by the equation $u = 2x$. **(a)** Determine the corresponding equation for the y component of velocity if $v = 0$ along the x axis. **(b)** For this flow field, what is the magnitude of the average velocity of the fluid crossing the surface OA of Fig. P6.23? Assume that the velocities are in feet per second when x and y are in feet.



■ **FIGURE P6.23**

6.24 The radial velocity component in an incompressible, two-dimensional flow field ($v_z = 0$) is

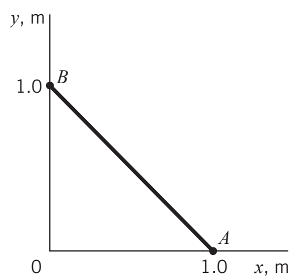
$$v_r = 2r + 3r^2\sin\theta$$

Determine the corresponding tangential velocity component, v_θ, required to satisfy conservation of mass.

6.25 The stream function for an incompressible flow field is given by the equation

$$\psi = 3x^2y - y^3$$

where the stream function has the units of m^2/s with x and y in meters. **(a)** Sketch the streamline(s) passing through the origin. **(b)** Determine the rate of flow across the straight path AB shown in Fig. P6.25.



■ **FIGURE P6.25**

6.26 The streamlines in a certain incompressible, two-dimensional flow field are all concentric circles so that $v_r = 0$. Determine the stream function for **(a)** $v_\theta = Ar$ and for **(b)** $v_\theta = Ar^{-1}$, where A is a constant.

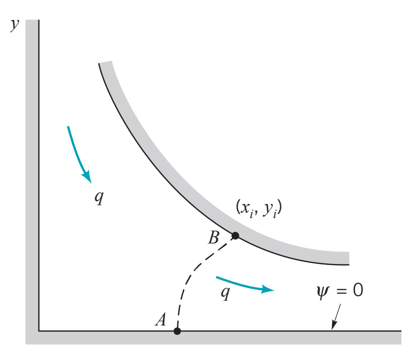
***6.27** The stream function for an incompressible, two-dimensional flow field is

$$\psi = 3x^2y + y$$

For this flow field, plot several streamlines.

6.28 Consider the incompressible, two-dimensional flow of a nonviscous fluid between the boundaries shown in Fig. P6.28. The velocity potential for this flow field is

$$\phi = x^2 - y^2$$



■ **FIGURE P6.28**

(a) Determine the corresponding stream function. **(b)** What is the relationship between the discharge, q, (per unit width normal to plane of paper) passing between the walls and the coordinates x_i, y_i of any point on the curved wall? Neglect body forces.

Section 6.3 Conservation of Linear Momentum

6.29 Obtain a photograph/image of a situation in which a fluid flow produces a force. Print this photo and write a brief paragraph that describes the situation involved.

Section 6.4 Inviscid Flow

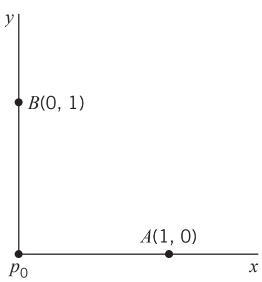
6.30 Obtain a photograph/image of a situation in which all or part of a flow field could be approximated by assuming inviscid flow. Print this photo and write a brief paragraph that describes the situation involved.

6.31 Given the streamfunction for a flow as $\psi = 4x^2 - 4y^2$, show that the Bernoulli equation can be applied between any two points in the flow field.

6.32 A two-dimensional flow field for a nonviscous, incompressible fluid is described by the velocity components

$$u = U_0 + 2y$$
$$v = 0$$

where U_0 is a constant. If the pressure at the origin (Fig. P6.32) is p_0, determine an expression for the pressure at **(a)** point A, and **(b)** point B. Explain clearly how you obtained your answer. Assume that the units are consistent and body forces may be neglected.



■ FIGURE P6.32

6.33 In a certain two-dimensional flow field, the velocity is constant with components $u = -4$ ft/s and $v = -2$ ft/s. Determine the corresponding stream function and velocity potential for this flow field. Sketch the equipotential line $\phi = 0$ which passes through the origin of the coordinate system.

6.34 The stream function for a given two-dimensional flow field is

$$\psi = 5x^2y - (5/3)y^3$$

Determine the corresponding velocity potential.

6.35 Determine the stream function corresponding to the velocity potential

$$\phi = x^3 - 3xy^2$$

Sketch the streamline $\psi = 0$, which passes through the origin.

6.36 A certain flow field is described by the stream function

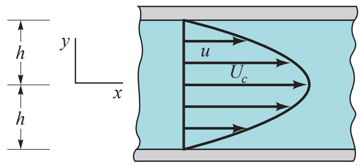
$$\psi = A\,\theta + B\,r\sin\theta$$

where A and B are positive constants. Determine the corresponding velocity potential and locate any stagnation points in this flow field.

6.37 It is known that the velocity distribution for two-dimensional flow of a viscous fluid between wide parallel plates (Fig. P6.37) is parabolic; that is,

$$u = U_c\left[1 - \left(\frac{y}{h}\right)^2\right]$$

with $v = 0$. Determine, if possible, the corresponding stream function and velocity potential.



■ FIGURE P6.37

6.38 The velocity potential for a certain inviscid flow field is

$$\phi = -(3x^2y - y^3)$$

where ϕ has the units of ft^2/s when x and y are in feet. Determine the pressure difference (in psi) between the points (1, 2) and (4, 4), where the coordinates are in feet, if the fluid is water and elevation changes are negligible.

6.39 The velocity potential for a flow is given by

$$\phi = \frac{a}{2}(x^2 - y^2)$$

where a is a constant. Determine the corresponding stream function and sketch the flow pattern.

6.40 The stream function for a two-dimensional, nonviscous, incompressible flow field is given by the expression

$$\psi = -2(x - y)$$

where the stream function has the units of ft^2/s with x and y in feet. **(a)** Is the continuity equation satisfied? **(b)** Is the flow field irrotational? If so, determine the corresponding velocity potential. **(c)** Determine the pressure gradient in the horizontal x direction at the point $x = 2$ ft, $y = 2$ ft.

6.41 The velocity potential for a certain inviscid, incompressible flow field is given by the equation

$$\phi = 2x^2y - (\tfrac{2}{3})y^3$$

where ϕ has the units of m^2/s when x and y are in meters. Determine the pressure at the point $x = 2$ m, $y = 2$ m if the pressure at $x = 1$ m, $y = 1$ m is 200 kPa. Elevation changes can be neglected, and the fluid is water.

6.42 A steady, uniform, incompressible, inviscid, two-dimensional flow makes an angle of 30° with the horizontal x axis. **(a)** Determine the velocity potential and the stream function for this flow. **(b)** Determine an expression for the pressure gradient in the vertical y direction. What is the physical interpretation of this result?

6.43 The streamlines for an incompressible, inviscid, two-dimensional flow field are all concentric circles, and the velocity varies directly with the distance from the common center of the streamlines; that is

$$v_\theta = Kr$$

where K is a constant. **(a)** For this *rotational* flow, determine, if possible, the stream function. **(b)** Can the pressure difference between the origin and any other point be determined from the Bernoulli equation? Explain.

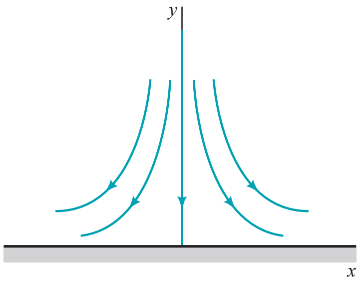
6.44 The velocity potential

$$\phi = -k(x^2 - y^2) \qquad (k = \text{constant})$$

may be used to represent the flow against an infinite plane boundary, as illustrated in Fig. P6.44. For flow in the vicinity of a stagnation point, it is frequently assumed that the pressure gradient along the surface is of the form

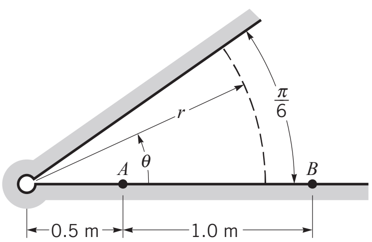
$$\frac{\partial p}{\partial x} = Ax$$

where A is a constant. Use the given velocity potential to show that this is true.



■ **FIGURE P6.44**

6.45 Water is flowing between wedge-shaped walls into a small opening as shown in Fig. P6.45. The velocity potential with units m^2/s for this flow is $\phi = -2 \ln r$ with r in meters. Determine the pressure differential between points A and B.

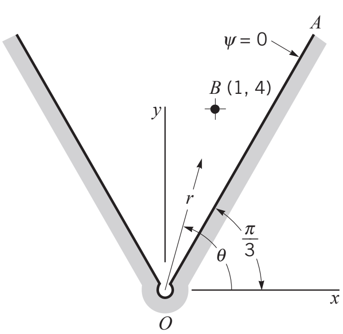


■ **FIGURE P6.45**

6.46 An ideal fluid flows between the inclined walls of a two-dimensional channel into a sink located at the origin (Fig. P6.46). The velocity potential for this flow field is

$$\phi = \frac{m}{2\pi} \ln r$$

where m is a constant. **(a)** Determine the corresponding stream function. Note that the value of the stream function along the wall OA is zero. **(b)** Determine the equation of the streamline passing through the point B, located at $x = 1, y = 4$.

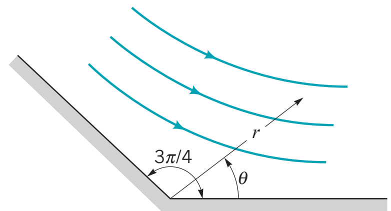


■ **FIGURE P6.46**

6.47 It is suggested that the velocity potential for the incompressible, nonviscous, two-dimensional flow along the wall shown in Fig. P6.47 is

$$\phi = r^{4/3} \cos \tfrac{4}{3}\theta$$

Is this a suitable velocity potential for flow along the wall? Explain.

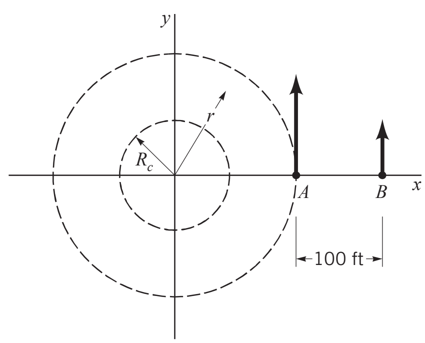


■ **FIGURE P6.47**

Section 6.5 Some Basic, Plane Potential Flows

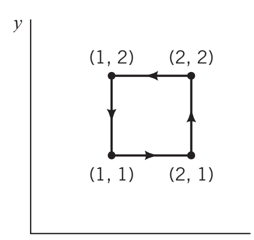
6.48 Obtain a photograph/image of a situation which approximates one of the basic, plane potential flows. Print this photo and write a brief paragraph that describes the situation involved.

6.49 As illustrated in Fig. P6.49, a tornado can be approximated by a free vortex of strength Γ for $r > R_c$, where R_c is the radius of the core. Velocity measurements at points A and B indicate that $V_A = 125$ ft/s and $V_B = 60$ ft/s. Determine the distance from point A to the center of the tornado. Why can the free vortex model not be used to approximate the tornado throughout the flow field ($r \geq 0$)?



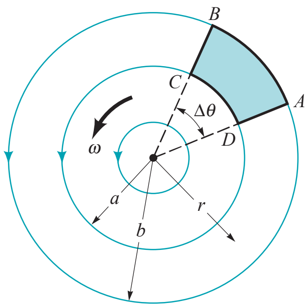
■ **FIGURE P6.49**

6.50 If the velocity field is given by $\mathbf{V} = ax\hat{\mathbf{i}} - ay\hat{\mathbf{j}}$, and a is a constant, find the circulation around the closed curve shown in Fig. P6.50.



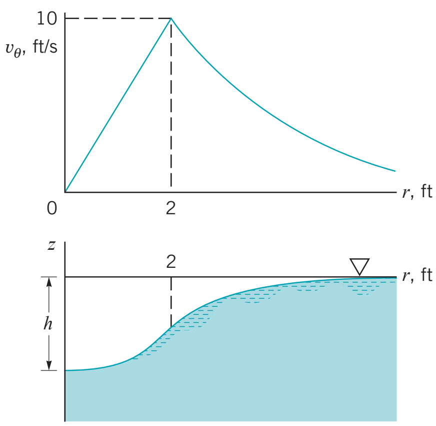
■ **FIGURE P6.50**

6.51 The streamlines in a particular two-dimensional flow field are all concentric circles, as shown in Fig. P6.51. The velocity is given by the equation $v_\theta = \omega r$ where ω is the angular velocity of the rotating mass of fluid. Determine the circulation around the path $ABCD$.



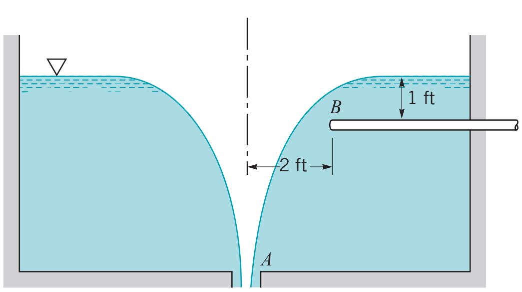
■ FIGURE P6.51

6.52 The motion of a liquid in an open tank is that of a combined vortex consisting of a forced vortex for $0 \leq r \leq 2$ ft and a free vortex for $r > 2$ ft. The velocity profile and the corresponding shape of the free surface are shown in Fig. P6.52. The free surface at the center of the tank is a depth h below the free surface at $r = \infty$. Determine the value of h. Note that $h = h_{\text{forced}} + h_{\text{free}}$, where h_{forced} and h_{free} are the corresponding depths for the forced vortex and the free vortex, respectively. (See Section 2.12.2 for further discussion regarding the forced vortex.)



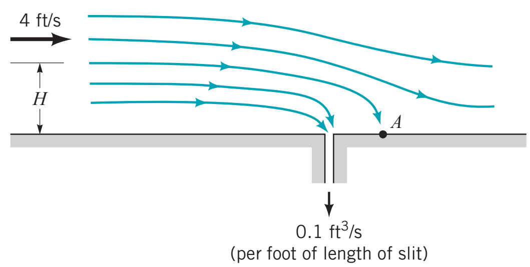
■ FIGURE P6.52

6.53 When water discharges from a tank through an opening in its bottom, a vortex may form with a curved surface profile, as shown in Fig. P6.53 and Video V6.4. Assume that the velocity distribution in the vortex is the same as that for a free vortex. At the same time the water is being discharged from the tank at point A, it is desired to discharge a small quantity of water through the pipe B. As the discharge through A is increased, the strength of the vortex, as indicated by its circulation, is increased. Determine the maximum strength that the vortex can have in order that no air is sucked in at B. Express your answer in terms of the circulation. Assume that the fluid level in the tank at a large distance from the opening at A remains constant and viscous effects are negligible.



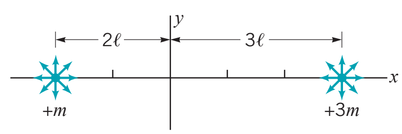
■ FIGURE P6.53

6.54 Water flows over a flat surface at 4 ft/s, as shown in Fig. P6.54. A pump draws off water through a narrow slit at a volume rate of 0.1 ft^3/s per foot length of the slit. Assume that the fluid is incompressible and inviscid and can be represented by the combination of a uniform flow and a sink. Locate the stagnation point on the wall (point A) and determine the equation for the stagnation streamline. How far above the surface, H, must the fluid be so that it does not get sucked into the slit?



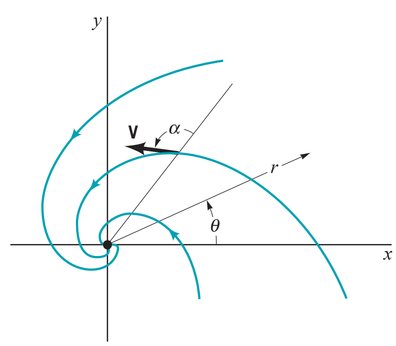
■ FIGURE P6.54

6.55 Two sources, one of strength m and the other with strength $3m$, are located on the x axis as shown in Fig. P6.55. Determine the location of the stagnation point in the flow produced by these sources.



■ FIGURE P6.55

6.56 The velocity potential for a spiral vortex flow is given by $\phi = (\Gamma/2\pi)\,\theta - (m/2\pi)\ln r$, where Γ and m are constants. Show that the angle, α, between the velocity vector and the radial direction is constant throughout the flow field (see Fig. P6.56).



■ FIGURE P6.56

6.57 For a free vortex (see Video V6.4) determine an expression for the pressure gradient **(a)** along a streamline, and **(b)** normal to a streamline. Assume that the streamline is in a horizontal plane, and express your answer in terms of the circulation.

6.58 (See Fluids in the News article titled "Some hurricanes facts," Section 6.5.3.) Consider a category five hurricane that has a maximum wind speed of 160 mph at the eye wall, 10 miles from the center of the hurricane. If the flow in the hurricane outside of the hurricane's eye is approximated as a free vortex, determine the wind speeds at locations 20 mi, 30 mi, and 40 mi from the center of the storm.

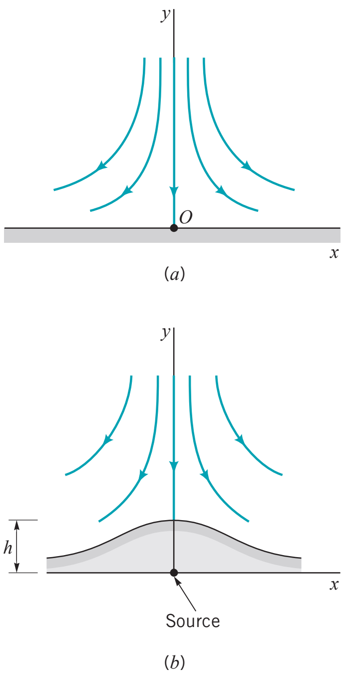
Section 6.6 Superposition of Basic, Plane Potential Flows

6.59 Obtain a photograph/image of a situation that mimics the superposition of potential flows (see Ex. 6.7). Print this photo and write a brief paragraph that describes the situation involved.

6.60 Potential flow against a flat plate (Fig. P6.60*a*) can be described with the stream function

$$\psi = Axy$$

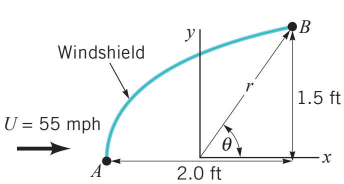
where A is a constant. This type of flow is commonly called a "stagnation point" flow since it can be used to describe the flow in the vicinity of the stagnation point at O. By adding a source of strength m at O, stagnation point flow against a flat plate with a "bump" is obtained as illustrated in Fig. P6.60*b*. Determine the relationship between the bump height, h, the constant, A, and the source strength, m.



■ FIGURE P6.60

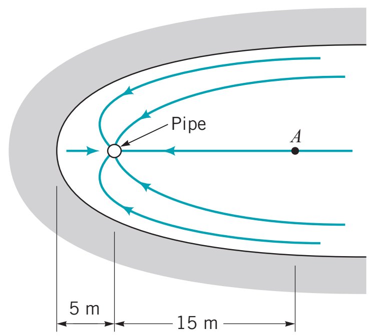
6.61 The combination of a uniform flow and a source can be used to describe flow around a streamlined body called a half-body. (See **Video V6.5.**) Assume that a certain body has the shape of a half-body with a thickness of 0.5 m. If this body is placed in an airstream moving at 15 m/s, what source strength is required to simulate flow around the body?

6.62 A vehicle windshield is to be shaped as a portion of a half-body with the dimensions shown in Fig. P6.62. **(a)** Make a scale drawing of the windshield shape. **(b)** For a free stream velocity of 55 mph, determine the velocity of the air at points A and B.



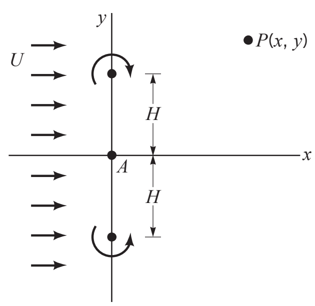
■ FIGURE P6.62

6.63 One end of a pond has a shoreline that resembles a half-body as shown in Fig. P6.63. A vertical porous pipe is located near the end of the pond so that water can be pumped out. When water is pumped at the rate of 0.08 m^3/s through a 3-m-long pipe, what will be the velocity at point A? *Hint:* Consider the flow *inside* a half-body. (See **Video V6.5.**)



■ FIGURE P6.63

6.64 Two free vortices of equal strength, but opposite direction of rotation, are superimposed with a uniform flow as shown in Fig. P6.64. The stream functions for these two vorticies are $\psi = -[\pm\Gamma/(2\pi)] \ln r$. **(a)** Develop an equation for the x-component of velocity, u, at point $P(x,y)$ in terms of Cartesian coordinates x and y. **(b)** Compute the x-component of velocity at point A and show that it depends on the ratio Γ/H.

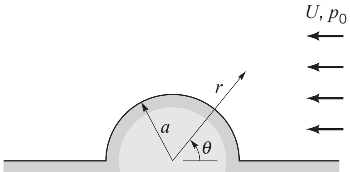


■ FIGURE P6.64

6.65 A Rankine oval is formed by combining a source–sink pair, each having a strength of 36 ft^2/s and separated by a distance of 12 ft along the x axis, with a uniform velocity of 10 ft/s (in the positive x direction). Determine the length and thickness of the oval.

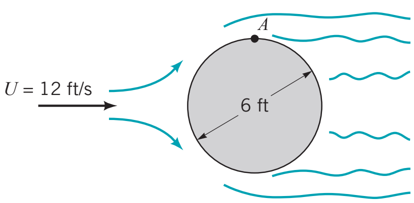
***6.66** Make use of Eqs. 6.107 and 6.109 to construct a table showing how ℓ/a, h/a, and ℓ/h for Rankine ovals depend on the parameter $\pi Ua/m$. Plot ℓ/h versus $\pi Ua/m$ and describe how this plot could be used to obtain the required values of m and a for a Rankine oval having a specific value of ℓ and h when placed in a uniform fluid stream of velocity, U.

6.67 An ideal fluid flows past an infinitely long, semicircular "hump" located along a plane boundary, as shown in Fig. P6.67. Far from the hump the velocity field is uniform, and the pressure is p_0. **(a)** Determine expressions for the maximum and minimum values of the pressure along the hump, and indicate where these points are located. Express your answer in terms of ρ, U, and p_0. **(b)** If the solid surface is the $\psi = 0$ streamline, determine the equation of the streamline passing through the point $\theta = \pi/2$, $r = 2a$.



■ FIGURE P6.67

6.68 Water flows around a 6-ft-diameter bridge pier with a velocity of 12 ft/s. Estimate the force (per unit length) that the water exerts on the pier. Assume that the flow can be approximated as an ideal fluid flow around the front half of the cylinder, but due to flow separation (see **Video V6.8**), the average pressure on the rear half is constant and approximately equal to ½ the pressure at point A (see Fig. P6.68).



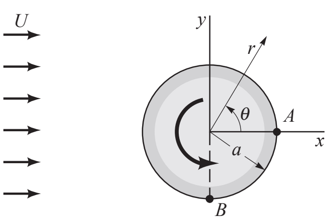
■ FIGURE P6.68

***6.69** Consider the steady potential flow around the circular cylinder shown in Fig. 6.26. On a plot show the variation of the magnitude of the dimensionless fluid velocity, V/U, along the positive y axis. At what distance, y/a (along the y axis), is the velocity within 1% of the free-stream velocity?

6.70 The velocity potential for a cylinder (Fig. P6.70) rotating in a uniform stream of fluid is

$$\phi = Ur\left(1 + \frac{a^2}{r^2}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta$$

where Γ is the circulation. For what value of the circulation will the stagnation point be located at: **(a)** point A, **(b)** point B?



■ FIGURE P6.70

6.71 Show that for a rotating cylinder in a uniform flow, the following pressure ratio equation is true.

$$\frac{p_{top} - p_{bottom}}{p_{stagnation}} = \frac{8q}{U}$$

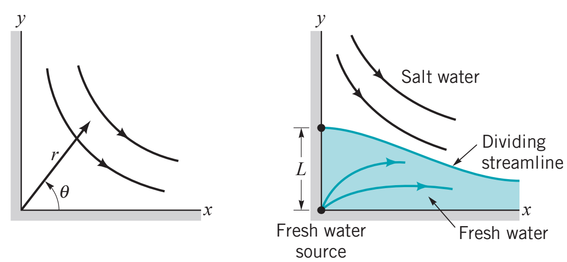
Here U is the velocity of the uniform flow and q is the surface speed of the rotating cylinder.

6.72 (See Fluids in the News article titled "A sailing ship without sails," Section 6.6.3.) Determine the magnitude of the total force developed by the two rotating cylinders on the Flettner "rotor-ship" due to the Magnus effect. Assume a wind speed relative to the ship of **(a)** 10 mph and **(b)** 30 mph. Each cylinder has a diameter of 9 ft, a length of 50 ft, and rotates at 750 rev/min. Use Eq. 6.124 and calculate the circulation by assuming the air sticks to the rotating cylinders. *Note*: This calculated force is at right angles to the direction of the wind and it is the component of this force in the direction of motion of the ship that gives the propulsive thrust. Also, due to viscous effects, the actual propulsive thrust will be smaller than that calculated from Eq. 6.124 which is based on inviscid flow theory.

6.73 A fixed circular cylinder of infinite length is placed in a steady, uniform stream of an incompressible, nonviscous fluid. Assume that the flow is irrotational. Prove that the drag on the cylinder is zero. Neglect body forces.

6.74 Repeat Problem 6.73 for a rotating cylinder for which the stream function and velocity potential are given by Eqs. 6.119 and 6.120, respectively. Verify that the lift is not zero and can be expressed by Eq. 6.124.

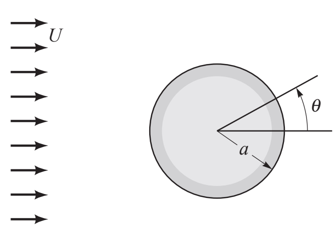
6.75 At a certain point at the beach, the coast line makes a right-angle bend, as shown in Fig. P6.75*a*. The flow of salt water in this bend can be approximated by the potential flow of an incompressible fluid in a right-angle corner. **(a)** Show that the stream function for this flow is $\psi = A\,r^2 \sin 2\theta$, where A is a positive constant. **(b)** A fresh-water reservoir is located in the corner. The salt water is to be kept away from the reservoir to avoid any possible seepage of salt water into the fresh water (Fig. P6.75*b*). The fresh-water source can be approximated as a line source having a strength m, where m is the volume rate of flow (per unit length) emanating from the source. Determine m if the salt water is not to get closer than a distance L to the corner. *Hint:* Find the value of m (in terms of A and L) so that a stagnation point occurs at $y = L$. **(c)** The streamline passing through the stagnation point would represent the line dividing the fresh water from the salt water. Plot this streamline.



(*a*) (*b*)

■ FIGURE P6.75

6.76 Typical inviscid flow solutions for flow around bodies indicate that the fluid flows smoothly around the body, even for blunt bodies as shown in **Video V6.10.** However, experience reveals that due to the presence of viscosity, the main flow may actually separate from the body creating a wake behind the body. As discussed in a later section (Section 9.2.6), whether or not separation takes place depends on the pressure gradient along the surface of the body, as calculated by inviscid flow theory. If the pressure decreases in the direction of flow (a *favorable* pressure gradient), no separation will occur. However, if the pressure increases in the direction of flow (an *adverse* pressure gradient), separation may occur. For the circular cylinder of Fig. P6.76 placed in a uniform stream with velocity, U,



■ FIGURE P6.76

determine an expression for the pressure gradient in the direction of flow on the surface of the cylinder. For what range of values for the angle θ will an adverse pressure gradient occur?

Section 6.8 Viscous Flow

6.77 Obtain a photograph/image of a situation in which the cylindrical form of the Navier–Stokes equations would be appropriate for the solution. Print this photo and write a brief paragraph that describes the situation involved.

6.78 For a steady, two-dimensional, incompressible flow, the velocity is given by $\mathbf{V} = (ax - cy)\hat{\mathbf{i}} + (-ay + cx)\hat{\mathbf{j}}$, where a and c are constants. Show that this flow can be considered inviscid.

6.79 Determine the shearing stress for an incompressible Newtonian fluid with a velocity distribution of $\mathbf{V} = (3xy^2 - 4x^3)\hat{\mathbf{i}} + (12x^2y - y^3)\hat{\mathbf{j}}$.

6.80 The two-dimensional velocity field for an incompressible Newtonian fluid is described by the relationship

$$\mathbf{V} = (12xy^2 - 6x^3)\hat{\mathbf{i}} + (18x^2y - 4y^3)\hat{\mathbf{j}}$$

where the velocity has units of m/s when x and y are in meters. Determine the stresses σ_{xx}, σ_{yy}, and τ_{xy} at the point $x = 0.5$ m, $y = 1.0$ m if pressure at this point is 6 kPa and the fluid is glycerin at 20 °C. Show these stresses on a sketch.

6.81 For a two-dimensional incompressible flow in the $x - y$ plane show that the z component of the vorticity, ζ_z, varies in accordance with the equation

$$\frac{D\zeta_z}{Dt} = \nu \nabla^2 \zeta_z$$

What is the physical interpretation of this equation for a nonviscous fluid? *Hint:* This *vorticity transport equation* can be derived from the Navier–Stokes equations by differentiating and eliminating the pressure between Eqs. 6.127a and 6.127b.

6.82 The velocity of a fluid particle moving along a horizontal streamline that coincides with the x axis in a plane, two-dimensional, incompressible flow field was experimentally found to be described by the equation $u = x^2$. Along this streamline determine an expression for **(a)** the rate of change of the v component of velocity with respect to y, **(b)** the acceleration of the particle, and **(c)** the pressure gradient in the x direction. The fluid is Newtonian.

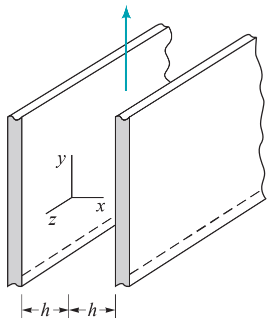
Section 6.9.1 Steady, Laminar Flow between Fixed Parallel Plates

6.83 Obtain a photograph/image of a situation which can be approximated by one of the simple cases covered in Sec. 6.9. Print this photo and write a brief paragraph that describes the situation involved.

6.84 Oil ($\mu = 0.4$ N · s/m^2) flows between two fixed horizontal infinite parallel plates with a spacing of 5 mm. The flow is laminar and steady with a pressure gradient of -900 (N/m^2) per unit meter. Determine the volume flowrate per unit width and the shear stress on the upper plate.

6.85 Two fixed, horizontal, parallel plates are spaced 0.4 in. apart. A viscous liquid ($\mu = 8 \times 10^{-3}$ lb · s/ft^2, $SG = 0.9$) flows between the plates with a mean velocity of 0.5 ft/s. The flow is laminar. Determine the pressure drop per unit length in the direction of flow. What is the maximum velocity in the channel?

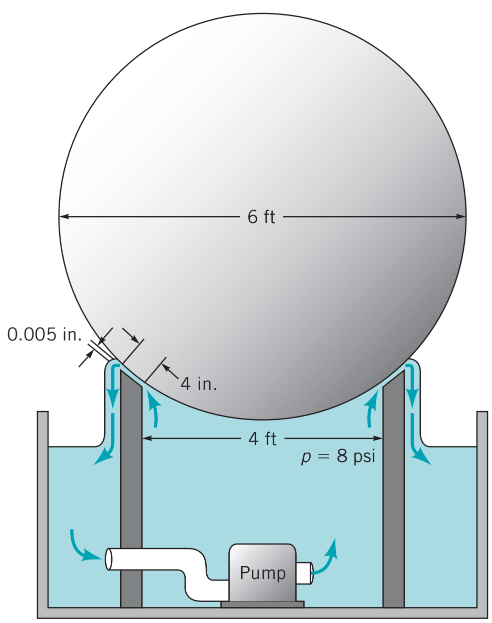
6.86 A viscous, incompressible fluid flows between the two infinite, vertical, parallel plates of Fig. P6.86. Determine, by use of the Navier–Stokes equations, an expression for the pressure gradient in the direction of flow. Express your answer in terms of the mean velocity. Assume that the flow is laminar, steady, and uniform.



■ FIGURE P6.86

6.87 A fluid is initially at rest between two horizontal, infinite, parallel plates. A constant pressure gradient in a direction parallel to the plates is suddenly applied and the fluid starts to move. Determine the appropriate differential equation(s), initial condition, and boundary conditions that govern this type of flow. You need not solve the equation(s).

6.88 (See Fluids in the News article titled "10 tons on 8 psi," Section 6.9.1.) A massive, precisely machined, 6-ft-diameter granite sphere rests upon a 4-ft-diameter cylindrical pedestal as shown in Fig. P6.88. When the pump is turned on and the water pressure within the pedestal reaches 8 psi, the sphere rises off the pedestal, creating a 0.005-in. gap through which the water flows. The sphere can then be rotated about any axis with minimal friction. **(a)** Estimate the pump flowrate, Q_0, required to accomplish this. Assume the flow in the gap between the sphere and the pedestal is essentially viscous flow between fixed, parallel plates. **(b)** Describe what would happen if the pump flowrate were increased to $2Q_0$.



■ FIGURE P6.88

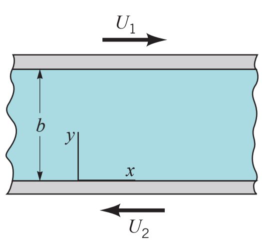
Section 6.9.2 Couette Flow

6.89 Two horizontal, infinite, parallel plates are spaced a distance b apart. A viscous liquid is contained between the plates. The bottom plate is fixed, and the upper plate moves parallel to the bottom plate with a velocity U. Because of the no-slip boundary condition (see **Video V6.11**), the liquid motion is caused by the liquid being dragged along by the moving boundary. There is no pressure gradient in the direction of flow. Note that this is a so-called simple *Couette flow* discussed in Section 6.9.2. **(a)** Start with the Navier–Stokes equations and determine the velocity distribution between the plates. **(b)** Determine an expression for the flowrate passing between the plates (for a unit width). Express your answer in terms of b and U.

6.90 A layer of viscous liquid of constant thickness (no velocity perpendicular to plate) flows steadily down an infinite, inclined plane. Determine, by means of the Navier–Stokes equations, the relationship between the thickness of the layer and the discharge per unit width. The flow is laminar, and assume air resistance is negligible so that the shearing stress at the free surface is zero.

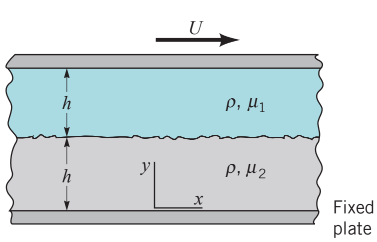
6.91 Due to the no-slip condition, as a solid is pulled out of a viscous liquid some of the liquid is also pulled along as described in Example 6.9 and shown in **Video V6.11.** Based on the results given in Example 6.9, show on a dimensionless plot the velocity distribution in the fluid film (v/V_0 vs. x/h) when the average film velocity, V, is 10% of the belt velocity, V_0.

6.92 An incompressible, viscous fluid is placed between horizontal, infinite, parallel plates as is shown in Fig. P6.92. The two plates move in opposite directions with constant velocities, U_1 and U_2, as shown. The pressure gradient in the x direction is zero, and the only body force is due to the fluid weight. Use the Navier–Stokes equations to derive an expression for the velocity distribution between the plates. Assume laminar flow.



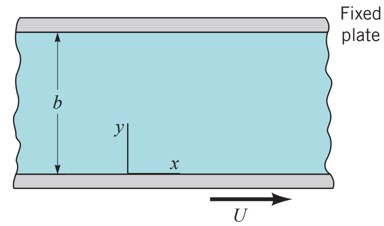
■ **FIGURE P6.92**

6.93 Two immiscible, incompressible, viscous fluids having the same densities but different viscosities are contained between two infinite, horizontal, parallel plates (Fig. P6.93). The bottom plate is fixed and the upper plate moves with a constant velocity U. Determine the velocity at the interface. Express your answer in terms of U, μ_1, and μ_2. The motion of the fluid is caused entirely by the movement of the upper plate; that is, there is no pressure gradient in the x direction. The fluid velocity and shearing stress are continuous across the interface between the two fluids. Assume laminar flow.



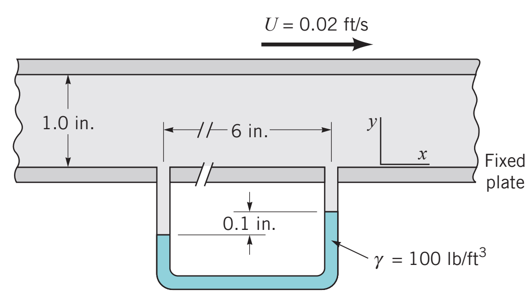
■ **FIGURE P6.93**

6.94 The viscous, incompressible flow between the parallel plates shown in Fig. P6.94 is caused by both the motion of the bottom plate and a pressure gradient, $\partial p/\partial x$. As noted in Section 6.9.2, an important dimensionless parameter for this type of problem is $P = -(b^2/2\,\mu U)\,(\partial p/\partial x)$ where μ is the fluid viscosity. Make a plot of the dimensionless velocity distribution (similar to that shown in Fig. 6.32*b*) for $P = 3$. For this case where does the maximum velocity occur?



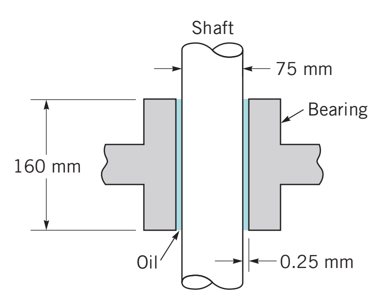
■ **FIGURE P6.94**

6.95 A viscous fluid (specific weight = 80 lb/ft^3; viscosity = 0.03 lb · s/ft^2) is contained between two infinite, horizontal parallel plates as shown in Fig. P6.95. The fluid moves between the plates under the action of a pressure gradient, and the upper plate moves with a velocity U while the bottom plate is fixed. A U-tube manometer connected between two points along the bottom indicates a differential reading of 0.1 in. If the upper plate moves with a velocity of 0.02 ft/s, at what distance from the bottom plate does the maximum velocity in the gap between the two plates occur? Assume laminar flow.



■ **FIGURE P6.95**

6.96 A vertical shaft passes through a bearing and is lubricated with an oil having a viscosity of 0.2 N · s/m^2 as shown in Fig. P6.96. Assume that the flow characteristics in the gap between the shaft and bearing are the same as those for laminar flow between infinite parallel plates with zero pressure gradient in the direction of flow. Estimate the torque required to overcome viscous resistance when the shaft is turning at 80 rev/min.



■ **FIGURE P6.96**

6.97 A viscous fluid is contained between two long concentric cylinders. The geometry of the system is such that the flow between the cylinders is approximately the same as the laminar flow between two infinite parallel plates. **(a)** Determine an expression for the torque required to rotate the outer cylinder with an angular velocity ω.

The inner cylinder is fixed. Express your answer in terms of the geometry of the system, the viscosity of the fluid, and the angular velocity. **(b)** For a small, rectangular element located at the fixed wall determine an expression for the rate of angular deformation of this element. (See **Video V6.3** and Fig. P6.9.)

***6.98** Oil (SAE 30) flows between parallel plates spaced 5 mm apart. The bottom plate is fixed, but the upper plate moves with a velocity of 0.2 m/s in the positive x direction. The pressure gradient is 60 kPa/m, and it is negative. Compute the velocity at various points across the channel and show the results on a plot. Assume laminar flow.

Section 6.9.3 Steady, Laminar Flow in Circular Tubes

6.99 Consider a steady, laminar flow through a straight horizontal tube having the constant elliptical cross section given by the equation

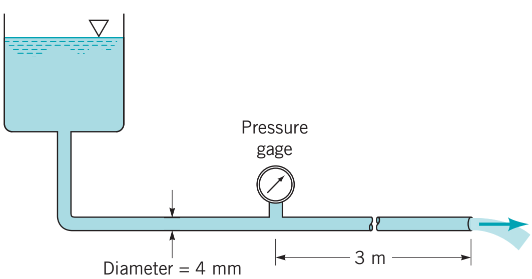
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The streamlines are all straight and parallel. Investigate the possibility of using an equation for the z component of velocity of the form

$$w = A\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

as an exact solution to this problem. With this velocity distribution, what is the relationship between the pressure gradient along the tube and the volume flowrate through the tube?

6.100 A simple flow system to be used for steady flow tests consists of a constant head tank connected to a length of 4-mm-diameter tubing as shown in Fig. P6.100. The liquid has a viscosity of 0.015 N · s/m^2, a density of 1200 kg/m^3, and discharges into the atmosphere with a mean velocity of 2 m/s. **(a)** Verify that the flow will be laminar. **(b)** The flow is fully developed in the last 3 m of the tube. What is the pressure at the pressure gage? **(c)** What is the magnitude of the wall shearing stress, τ_{rz}, in the fully developed region?



■ **FIGURE P6.100**

6.101 **(a)** Show that for Poiseuille flow in a tube of radius R the magnitude of the wall shearing stress, τ_{rz}, can be obtained from the relationship

$$|(\tau_{rz})_{\text{wall}}| = \frac{4\mu Q}{\pi R^3}$$

for a Newtonian fluid of viscosity μ. The volume rate of flow is Q. **(b)** Determine the magnitude of the wall shearing stress for a fluid having a viscosity of 0.004 N · s/m^2 flowing with an average velocity of 130 mm/s in a 2-mm-diameter tube.

6.102 An infinitely long, solid, vertical cylinder of radius R is located in an infinite mass of an incompressible fluid. Start with the Navier–Stokes equation in the θ direction and derive an expression for the velocity distribution for the steady flow case in which the cylinder is rotating about a fixed axis with a constant angular velocity ω. You need not consider body forces. Assume that the flow is axisymmetric and the fluid is at rest at infinity.

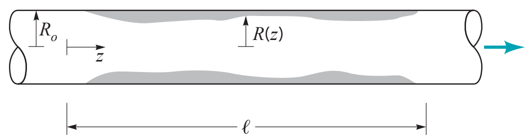
***6.103** As is shown by Eq. 6.150 the pressure gradient for laminar flow through a tube of constant radius is given by the expression

$$\frac{\partial p}{\partial z} = -\frac{8\mu Q}{\pi R^4}$$

For a tube whose radius is changing very gradually, such as the one illustrated in Fig. P6.103, it is expected that this equation can be used to approximate the pressure change along the tube if the actual radius, $R(z)$, is used at each cross section. The following measurements were obtained along a particular tube.

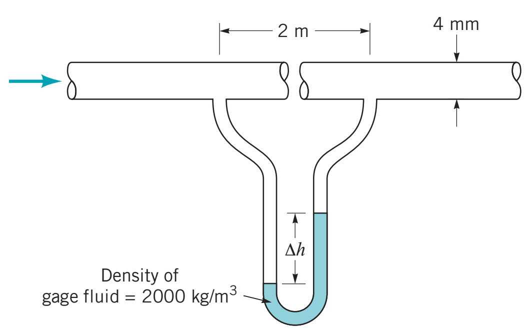
z/ℓ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$R(z)/R_o$	1.00	0.73	0.67	0.65	0.67	0.80	0.80	0.71	0.73	0.77	1.00

Compare the pressure drop over the length ℓ for this nonuniform tube with one having the constant radius R_o. *Hint:* To solve this problem you will need to numerically integrate the equation for the pressure gradient given above.



■ **FIGURE P6.103**

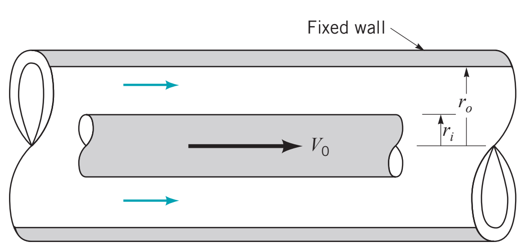
6.104 A liquid (viscosity = 0.002 N·s/m^2; density = 1000 kg/m^3) is forced through the circular tube shown in Fig. P6.104. A differential manometer is connected to the tube as shown to measure the pressure drop along the tube. When the differential reading, Δh, is 9 mm, what is the mean velocity in the tube?



■ **FIGURE P6.104**

Section 6.9.4 Steady, Axial, Laminar Flow in an Annulus

6.105 An incompressible Newtonian fluid flows steadily between two infinitely long, concentric cylinders as shown in Fig. P6.105. The outer cylinder is fixed, but the inner cylinder moves with a longitudinal velocity V_0 as shown. The pressure gradient in the axial direction is $-\Delta p/\ell$. For what value of V_0 will the drag on the inner cylinder be zero? Assume that the flow is laminar, axisymmetric, and fully developed.



■ **FIGURE P6.105**

6.106 A viscous fluid is contained between two infinitely long, vertical, concentric cylinders. The outer cylinder has a radius r_o and rotates with an angular velocity ω. The inner cylinder is fixed and has a radius r_i. Make use of the Navier–Stokes equations to obtain an exact solution for the velocity distribution in the gap. Assume that the flow in the gap is axisymmetric (neither velocity nor pressure are functions of angular position θ within the gap) and that there are no velocity components other than the tangential component. The only body force is the weight.

6.107 For flow between concentric cylinders, with the outer cylinder rotating at an angular velocity ω and the inner cylinder fixed, it is commonly assumed that the tangential velocity (v_θ) distribution in the gap between the cylinders is linear. Based on the exact solution to this problem (see Problem 6.106) the velocity distribution in the gap is not linear. For an outer cylinder with radius $r_o = 2.00$ in. and an inner cylinder with radius $r_i = 1.80$ in., show, with the aid of a plot, how the dimensionless velocity distribution, $v_\theta/r_o\omega$, varies with the dimensionless radial position, r/r_o, for the exact and approximate solutions.

6.108 A viscous liquid ($\mu = 0.012 \text{ lb} \cdot \text{s/ft}^2$, $\rho = 1.79 \text{ slugs/ft}^3$) flows through the annular space between two horizontal, fixed, concentric cylinders. If the radius of the inner cylinder is 1.5 in. and the radius of the outer cylinder is 2.5 in., what is the pressure drop along the axis of the annulus per foot when the volume flowrate is 0.14 ft^3/s?

6.109 Show how Eq. 6.155 is obtained.

6.110 A wire of diameter d is stretched along the centerline of a pipe of diameter D. For a given pressure drop per unit length of pipe, by how much does the presence of the wire reduce the flowrate if **(a)** $d/D = 0.1$; **(b)** $d/D = 0.01$?

Section 6.10 Other Aspects of Differential Analysis

6.111 Obtain a photograph/image of a situation in which CFD has been used to solve a fluid flow problem. Print this photo and write a brief paragraph that describes the situation involved.

■ Life Long Learning Problems

6.112 What sometimes appear at first glance to be simple fluid flows can contain subtle, complex fluid mechanics. One such example is the stirring of tea leaves in a teacup. Obtain information about "Einstein's tea leaves" and investigate some of the complex fluid motions interacting with the leaves. Summarize your findings in a brief report.

6.113 Computational fluid dynamics (CFD) has moved from a research tool to a design tool for engineering. Initially, much of the work in CFD was focused in the aerospace industry, but now has expanded into other areas. Obtain information on what other industries (e.g., automotive) make use of CFD in their engineering design. Summarize your findings in a brief report.

■ FE Exam Problems

Sample FE (Fundamentals of Engineering) exam questions for fluid mechanics are provided on the book's web site, www.wiley.com/college/munson.